



HUMAN CAPITAL
NATIONAL COHESION STRATEGY



Wrocław University of Technology

EUROPEAN UNION
EUROPEAN
SOCIAL FUND



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Dynamics of Structures

Part 6

Zbigniew Wójcicki Jacek Grosel

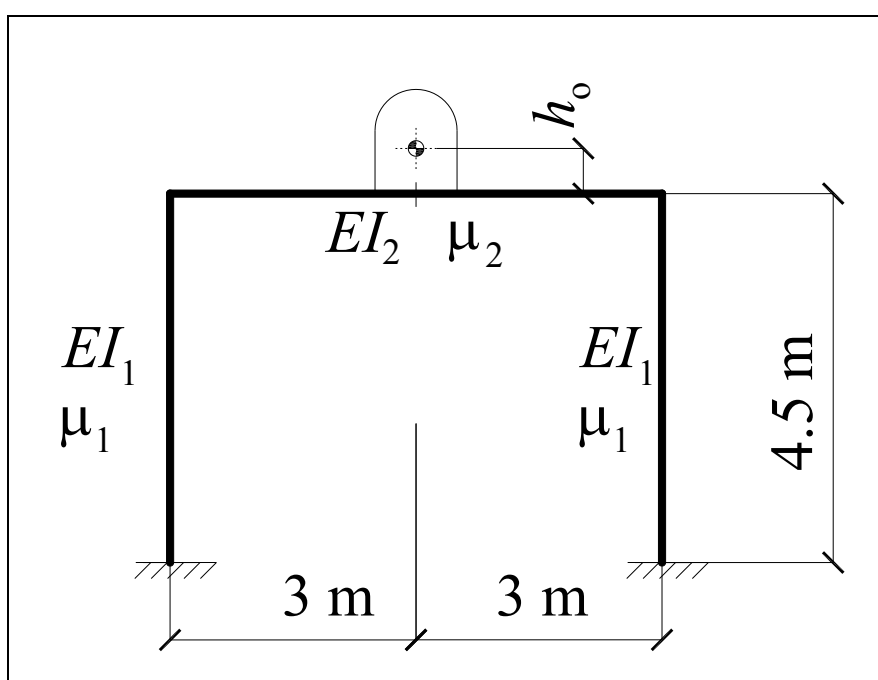
Multi-Degree-of-Freedom Systems (MDOF)

Dynamic Analysis of Beams and Frames Using Lumped Mass Models

Illustrative example 2

Dynamical analysis of a steel one-bay frame which supports a rotating machine.

Dynamic scheme (diagram) of the frame



Data:

$$EA = \infty$$

$$EJ_1 = 12 \text{ MNm}^2$$

$$m_1 = 60 \text{ kg/m}$$

$$G = 500 \text{ kg}$$

$$e = 2 \text{ mm}$$

$$\gamma = 2\alpha = 0.2$$

$$EJ_2 = 18 \text{ MNm}^2$$

$$m_2 = 40 \text{ kg/m}$$

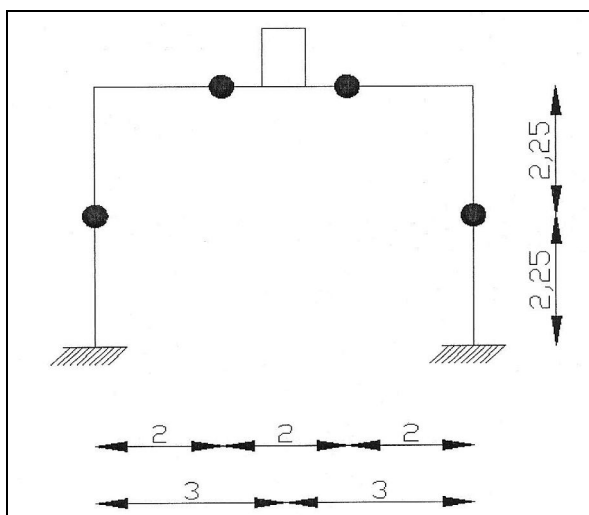
$$G_w = 200 \text{ kg}$$

$$n = 400 \text{ cycle/min}$$

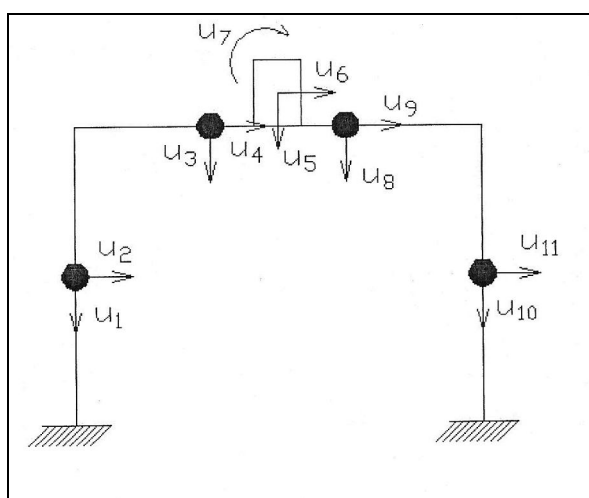
THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Selection of the dynamic scheme:

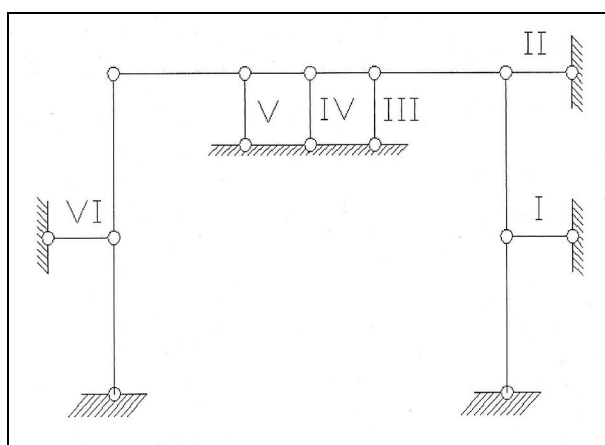
Lumped mass model – Variant 1



Local coordinates



Kinematic chain

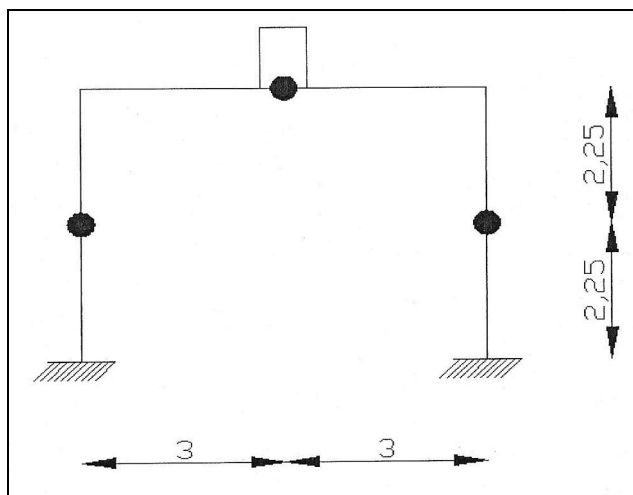


THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

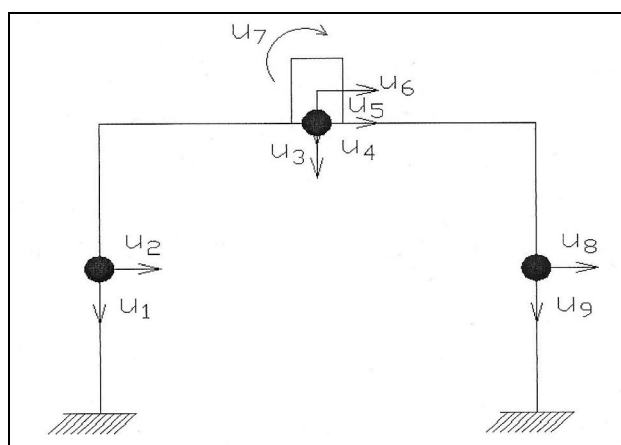
Number of degrees of freedom

$$d = d_{\Delta} + d_{\varphi} = 5 + 1 = 6$$

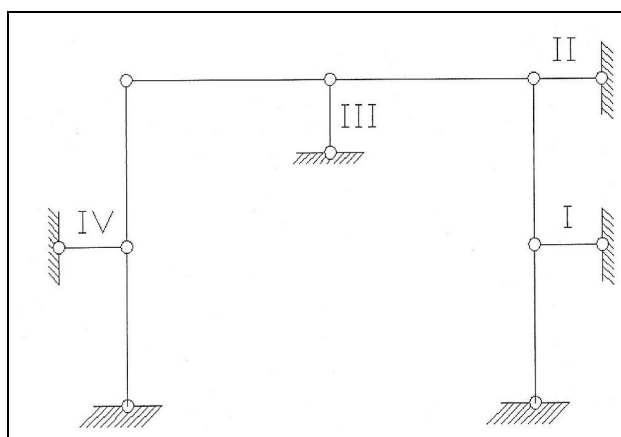
Lumped mass model – Variant 2



Local coordinates



Kinematic chain



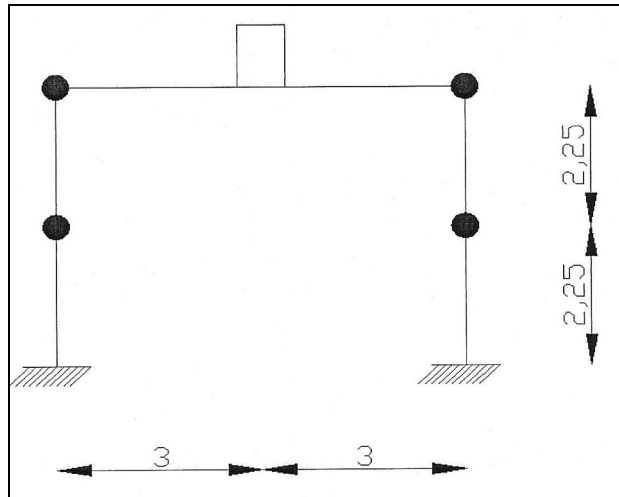


THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

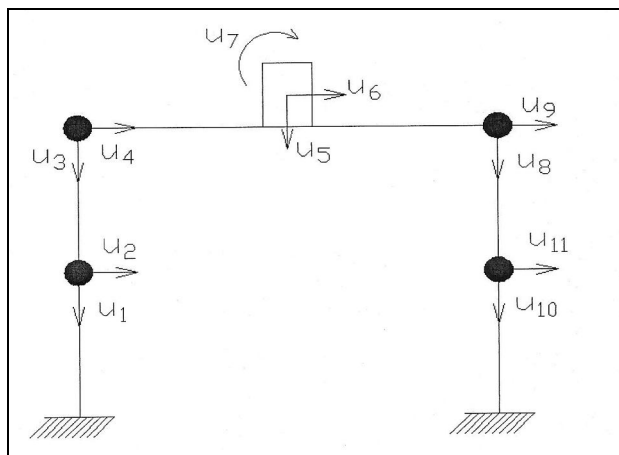
Number of degrees of freedom

$$d = d_{\Delta} + d_{\varphi} = 4 + 1 = 5$$

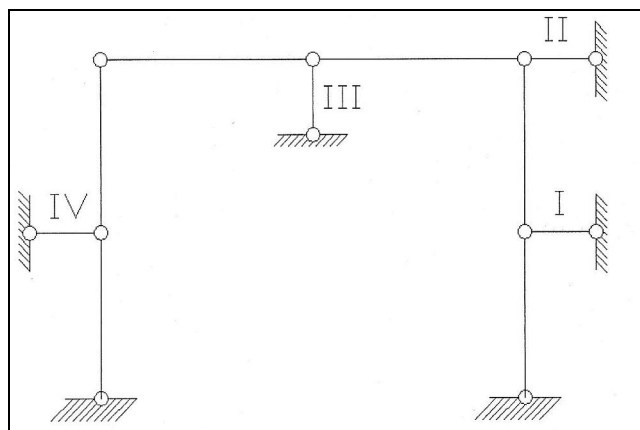
Lumped mass model – Variant 3



Local coordinates:



Kinematic chain



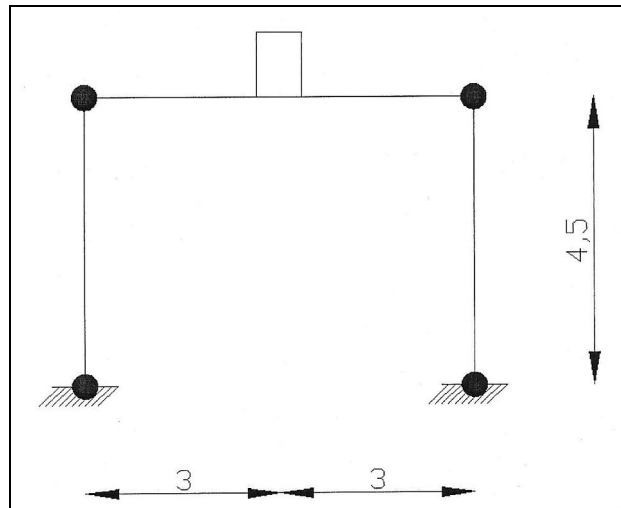


THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

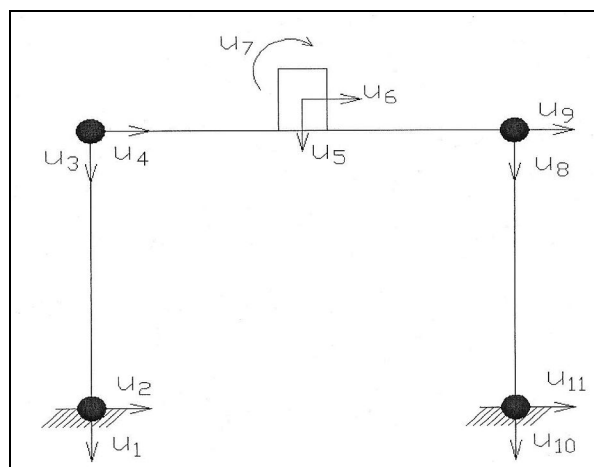
Number of degrees of freedom

$$d = d_{\Delta} + d_{\varphi} = 4 + 1 = 5$$

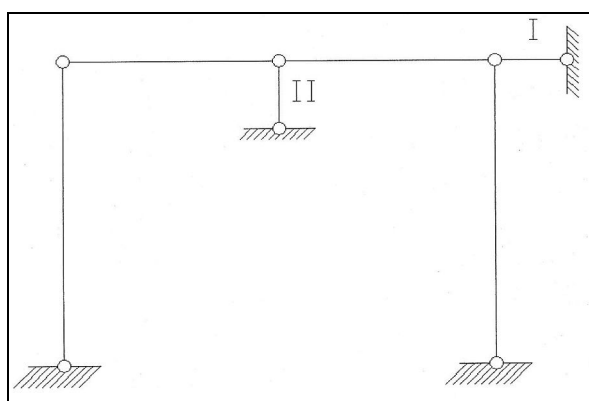
Lumped mass model – Variant 4



Local coordinates



Kinematic chain

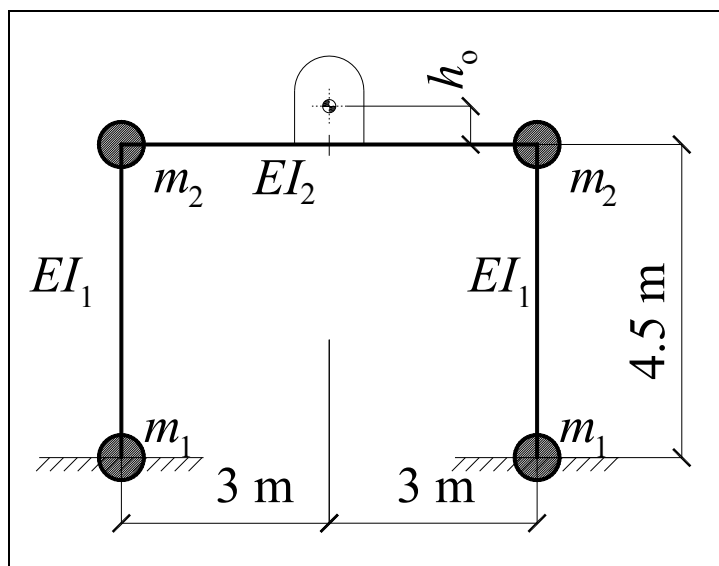


THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Number of degrees of freedom

$$d = d_{\Delta} + d_{\varphi} = 2 + 1 = 3$$

The variant chosen for further analysis



$$m_1 = \frac{1}{2} \cdot 60 \frac{\text{kg}}{\text{m}} \cdot 4.5 \text{ m} = 135 \text{ kg}$$

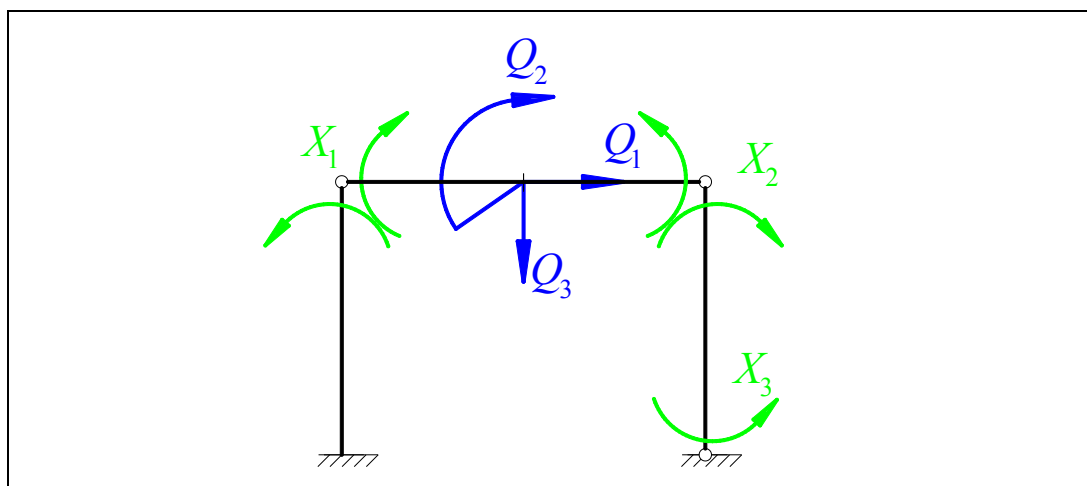
$$m_2 = \frac{1}{2} \cdot 60 \frac{\text{kg}}{\text{m}} \cdot 4.5 \text{ m} + \frac{1}{2} \cdot 40 \frac{\text{kg}}{\text{m}} \cdot 6 \text{ m} = 255 \text{ kg}$$

$$G = 500 \text{ kg}$$

Selection of the method of solving the problem

Primary Structure of Force Method

$$n_h = e - 3t = 6 - 3 \cdot 1 = 3$$

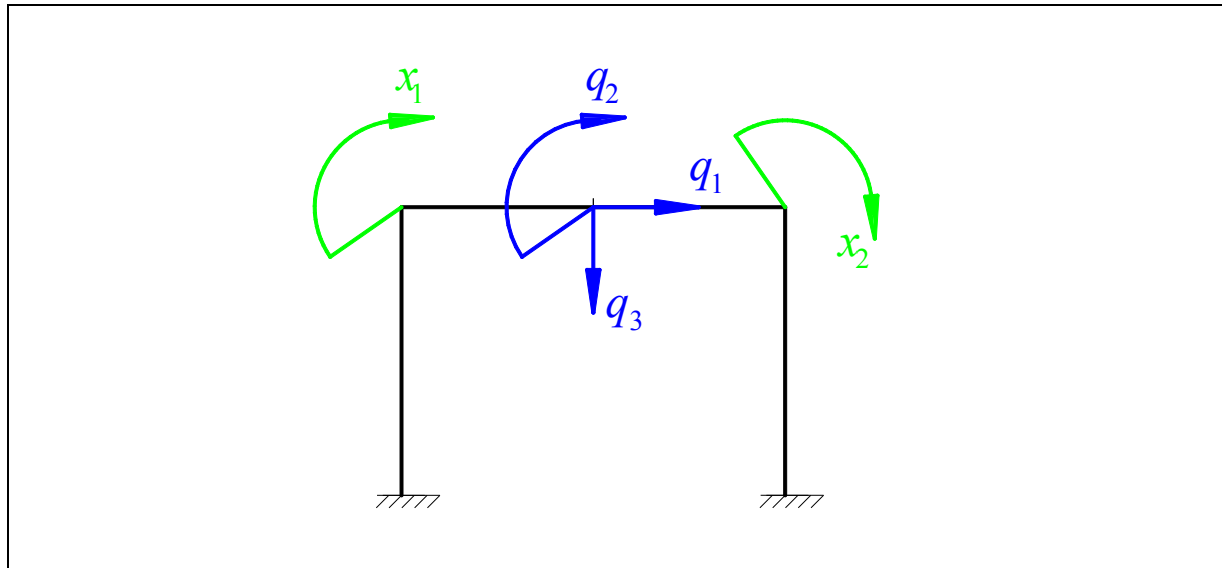


THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Displacement Method

$$n_g = n_\Delta + n_\varphi = 3 + 2 = 5$$

$$n_{gd} = n_g - d = 5 - 3 = 2$$



Since $n_{gd} < n_h$ - the displacement method is chosen

Determination of stiffness and flexibility matrices

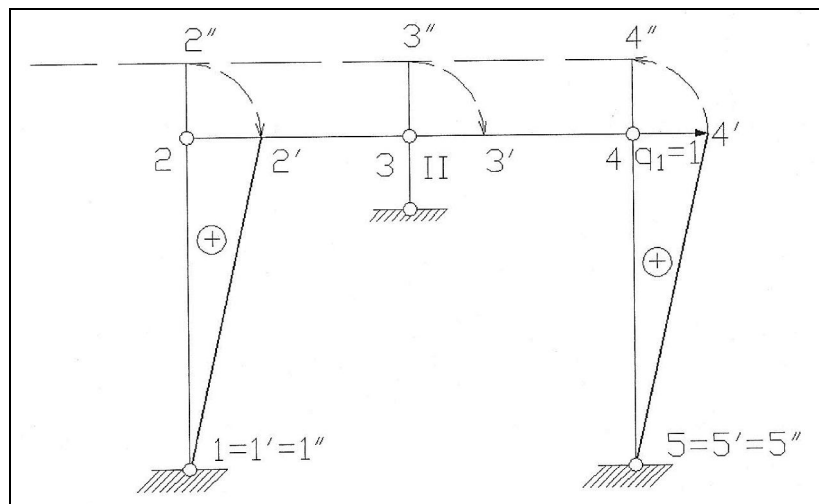
$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ x_1 \\ x_2 \end{bmatrix} \quad \hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_{qq} & \mathbf{K}_{qx} \\ \mathbf{K}_{xq} & \mathbf{K}_{xx} \end{bmatrix}$$

$$\mathbf{K}_{qq} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \quad \mathbf{K}_{qx} = \begin{bmatrix} k_{1x1} & k_{1x2} \\ k_{2x1} & k_{2x2} \\ k_{3x1} & k_{3x2} \end{bmatrix},$$

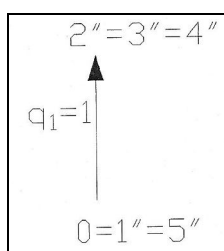
$$\mathbf{K}_{xq} = \begin{bmatrix} k_{x11} & k_{x12} & k_{x13} \\ k_{x21} & k_{x22} & k_{x23} \end{bmatrix}, \quad \mathbf{K}_{xx} = \begin{bmatrix} k_{x1x1} & k_{x1x2} \\ k_{x2x1} & k_{x2x2} \end{bmatrix}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Unit displacement $q_1 = 1$



Turned displacements pole plan

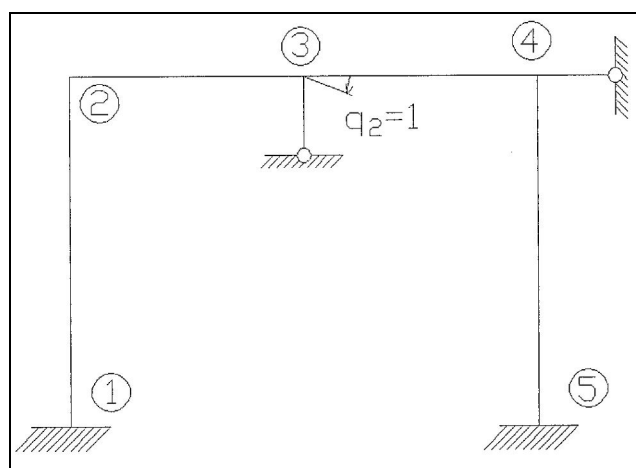


$$\psi_{12} = \psi_{45} = \frac{1}{4.5\text{m}} = \frac{2}{9\text{m}}$$

$$M_{12} = M_{21} = M_{45} = M_{54} = -6 \cdot \frac{EI_1}{4.5\text{m}} \cdot \frac{2}{9\text{m}} = -6 \cdot \frac{12\text{MNm}^2}{4.5\text{m}} \cdot \frac{2}{9\text{m}} = -3.556\text{MN}$$

$$M_{32} = M_{23} = M_{43} = M_{34} = 0$$

Unit displacement $q_2 = 1$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

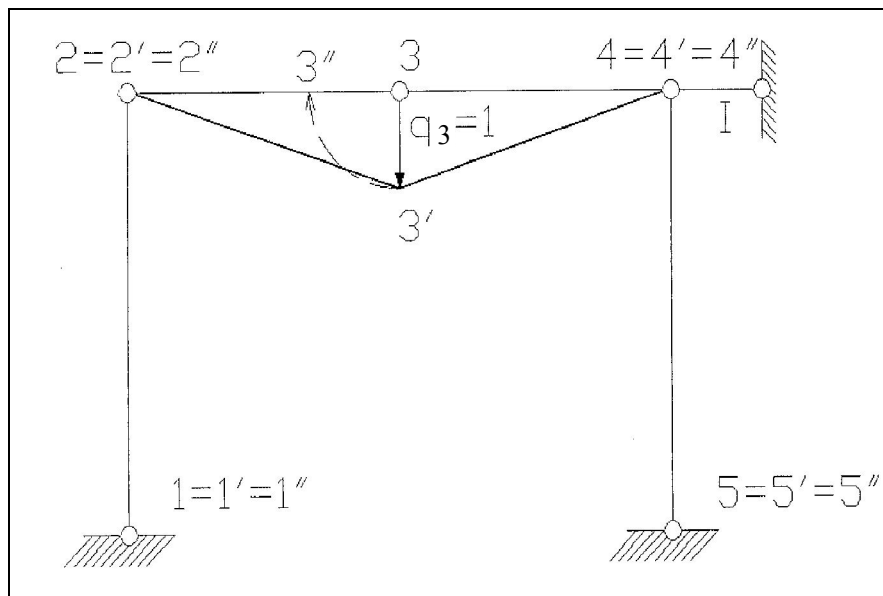
$$M_{23} = 2 \cdot \frac{EI_2}{3m} = 2 \cdot \frac{18\text{MNm}^2}{3m} = 12\text{MNm}$$

$$M_{32} = 4 \cdot \frac{EI_2}{3m} = 4 \cdot \frac{18\text{MNm}^2}{3m} = 24\text{MNm}$$

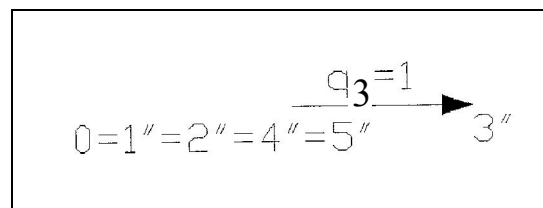
$$M_{34} = 4 \cdot \frac{EI_2}{3m} = 4 \cdot \frac{18\text{MNm}^2}{3m} = 24\text{MNm}$$

$$M_{32} = 2 \cdot \frac{EI_2}{3m} = 2 \cdot \frac{18\text{MNm}^2}{3m} = 12\text{MNm}$$

Unit displacement $q_3 = 1$



Turned displacements pole plan



$$\psi_{23} = \frac{1}{3m} = \frac{1}{3} \frac{1}{m}$$

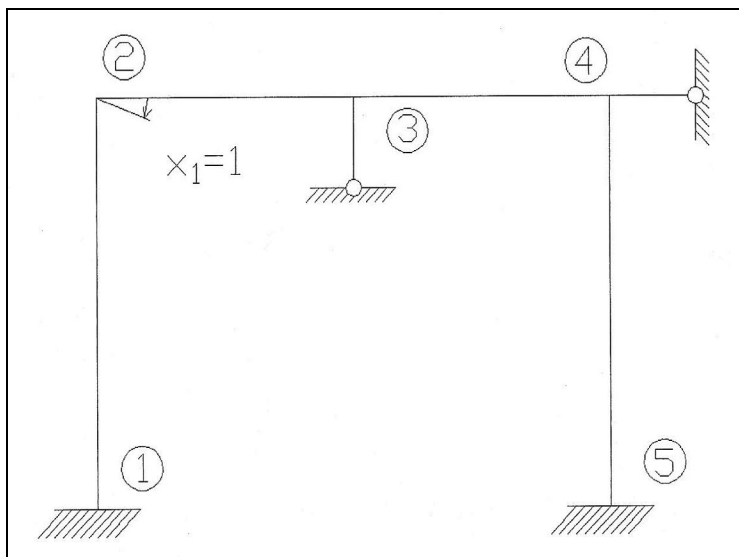
$$\psi_{34} = -\frac{1}{3m} = -\frac{1}{3} \frac{1}{m}$$

$$M_{23} = M_{32} = -6 \cdot \frac{EI_2}{3m} \cdot \frac{1}{3m} = -6 \cdot \frac{18\text{MNm}^2}{3m} \cdot \frac{1}{3} \frac{1}{m} = -12\text{MN}$$

$$M_{34} = M_{43} = -6 \cdot \frac{EI_2}{3m} \cdot \left(-\frac{1}{3m}\right) = -6 \cdot \frac{18\text{MNm}^2}{3m} \cdot \left(-\frac{1}{3} \frac{1}{m}\right) = 12\text{MN}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Unit displacement $x_1 = 1$



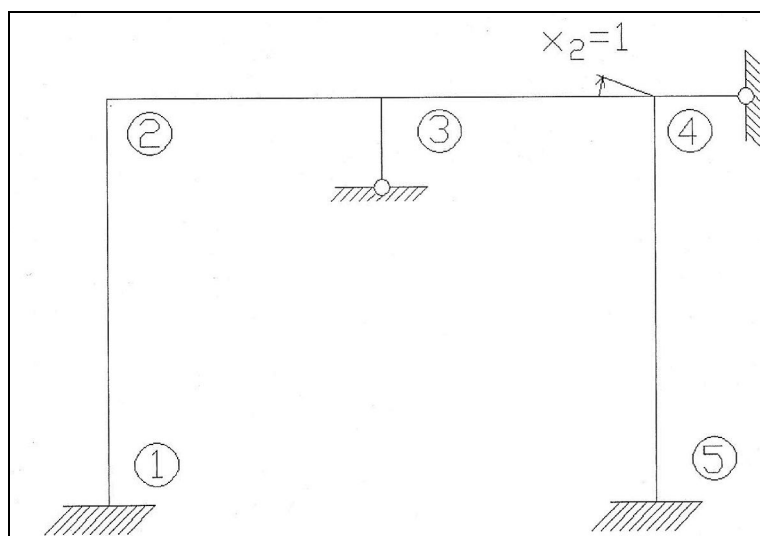
$$M_{12} = 2 \cdot \frac{EI_1}{4.5\text{m}} = 2 \cdot \frac{12\text{MNm}^2}{4.5\text{m}} = 5.333\text{MNm}$$

$$M_{21} = 4 \cdot \frac{EI_1}{4.5\text{m}} = 4 \cdot \frac{12\text{MNm}^2}{4.5\text{m}} = 10.667\text{MNm}$$

$$M_{23} = 4 \cdot \frac{EI_2}{3\text{m}} = 4 \cdot \frac{18\text{MNm}^2}{3\text{m}} = 24\text{MNm}$$

$$M_{32} = 2 \cdot \frac{EI_2}{3\text{m}} = 2 \cdot \frac{18\text{MNm}^2}{3\text{m}} = 12\text{MNm}$$

Unit displacement $x_2 = 1$





THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$M_{34} = 2 \cdot \frac{EI_2}{3m} = 2 \cdot \frac{18 \text{ MNm}^2}{3m} = 12 \text{ MNm}$$

$$M_{43} = 4 \cdot \frac{EI_2}{3m} = 4 \cdot \frac{18 \text{ MNm}^2}{3m} = 24 \text{ MNm}$$

$$M_{45} = 4 \cdot \frac{EI_1}{4.5m} = 4 \cdot \frac{12 \text{ MNm}^2}{4.5m} = 10.667 \text{ MNm}$$

$$M_{54} = 2 \cdot \frac{EI_1}{4.5m} = 2 \cdot \frac{12 \text{ MNm}^2}{4.5m} = 5.333 \text{ MNm}$$

The stiffness matrix elements determination

$$k_{11} = 2(-2)(-3.556 \text{ MN}) \frac{2}{9m} = 3.161 \frac{\text{MN}}{m}$$

$$k_{12} = k_{21} = 0$$

$$k_{13} = k_{31} = 0$$

$$k_{1x1} = k_{x11} = -(5.333 \text{ MNm} + 10.667 \text{ MNm}) \frac{2}{9m} = -3.556 \text{ MN}$$

$$k_{1x2} = k_{x21} = -(5.333 \text{ MNm} + 10.667 \text{ MNm}) \frac{2}{9m} = -3.556 \text{ MN}$$

$$k_{22} = 2 \cdot 24 \text{ MNm} = 48 \text{ MNm}$$

$$k_{23} = k_{32} = -12 + 12 \text{ MN} = 0$$

$$k_{2x1} = k_{x12} = 12 \text{ MNm}$$

$$k_{2x2} = k_{x22} = 12 \text{ MNm}$$

$$k_{33} = -(-2 \cdot 12 \text{ MN}) \frac{1}{3m} - 2 \cdot 12 \text{ MN} \frac{-1}{3m} = 16 \frac{\text{MN}}{m}$$

$$k_{3x1} = k_{x13} = -(12 \text{ MNm} + 24 \text{ MNm}) \frac{1}{3m} = -12 \text{ MN}$$

$$k_{3x2} = k_{x23} = -(12 \text{ MNm} + 24 \text{ MNm}) \frac{-1}{3m} = 12 \text{ MN}$$

$$k_{x1x1} = 24 \text{ MNm} = 48 \text{ MNm} + 10.667 \text{ MNm} = 34.667 \text{ MNm}$$

$$k_{x1x2} = k_{x2x1} = 0$$

$$k_{x2x2} = 24 \text{ MNm} + 10.667 \text{ MNm} = 34.667 \text{ MNm}$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} 3.161 \frac{\text{MN}}{m} & 0 & 0 & -3.556 \text{ MN} & -3.556 \text{ MN} \\ 0 & 48 \text{ MNm} & 0 & 12 \text{ MNm} & 12 \text{ MNm} \\ 0 & 0 & 16 \frac{\text{MN}}{m} & -12 \text{ MNm} & 12 \text{ MNm} \\ -3.556 \text{ MN} & 12 \text{ MNm} & -12 \text{ MNm} & 34.667 \text{ MNm} & 0 \\ -3.556 \text{ MN} & 12 \text{ MNm} & 12 \text{ MNm} & 0 & 34.667 \text{ MNm} \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Dynamic condensation

$$\mathbf{K}_{qq} = \begin{bmatrix} 3.161 \frac{\text{MN}}{\text{m}} & 0 & 0 \\ 0 & 48 \text{MNm} & 0 \\ 0 & 0 & 16 \frac{\text{MN}}{\text{m}} \end{bmatrix}$$

$$\mathbf{K}_{qx} = \begin{bmatrix} -3.556 \text{MN} & -3.556 \text{MN} \\ 12 \text{MNm} & 12 \text{MNm} \\ -12 \text{MN} & 12 \text{MN} \end{bmatrix}$$

$$\mathbf{K}_{xq} = \begin{bmatrix} -3.556 \text{MN} & 12 \text{MNm} & -12 \text{MN} \\ -3.556 \text{MN} & 12 \text{MNm} & 12 \text{MN} \end{bmatrix}$$

$$\mathbf{K}_{xx} = \begin{bmatrix} 34,667 \text{MNm} & 0 \\ 0 & 34,667 \text{MNm} \end{bmatrix}$$

$$\begin{aligned} \mathbf{K} = \mathbf{K}_{qq} - \mathbf{K}_{qx} \cdot \mathbf{K}_{xx}^{-1} \cdot \mathbf{K}_{xq} &= \begin{bmatrix} 3.161 \frac{\text{MN}}{\text{m}} & 0 & 0 \\ 0 & 48 \text{MNm} & 0 \\ 0 & 0 & 16 \frac{\text{MN}}{\text{m}} \end{bmatrix} + \\ &- \begin{bmatrix} -3.556 \text{MN} & -3.556 \text{MN} \\ 12 \text{MNm} & 12 \text{MNm} \\ -12 \text{MN} & 12 \text{MN} \end{bmatrix} \cdot \begin{bmatrix} 34,667 \text{MNm} & 0 \\ 0 & 34,667 \text{MNm} \end{bmatrix}^{-1} \cdot \begin{bmatrix} -3.556 \text{MN} & 12 \text{MNm} & -12 \text{MN} \\ -3.556 \text{MN} & 12 \text{MNm} & 12 \text{MN} \end{bmatrix} \\ \mathbf{K} &= \begin{bmatrix} 2.43115 \frac{\text{MN}}{\text{m}} & 2.46154 \text{MN} & 0 \\ 2.46154 \text{MN} & 39.6923 \text{MNm} & 0 \\ 0 & 0 & 7.69231 \frac{\text{MN}}{\text{m}} \end{bmatrix} \end{aligned}$$

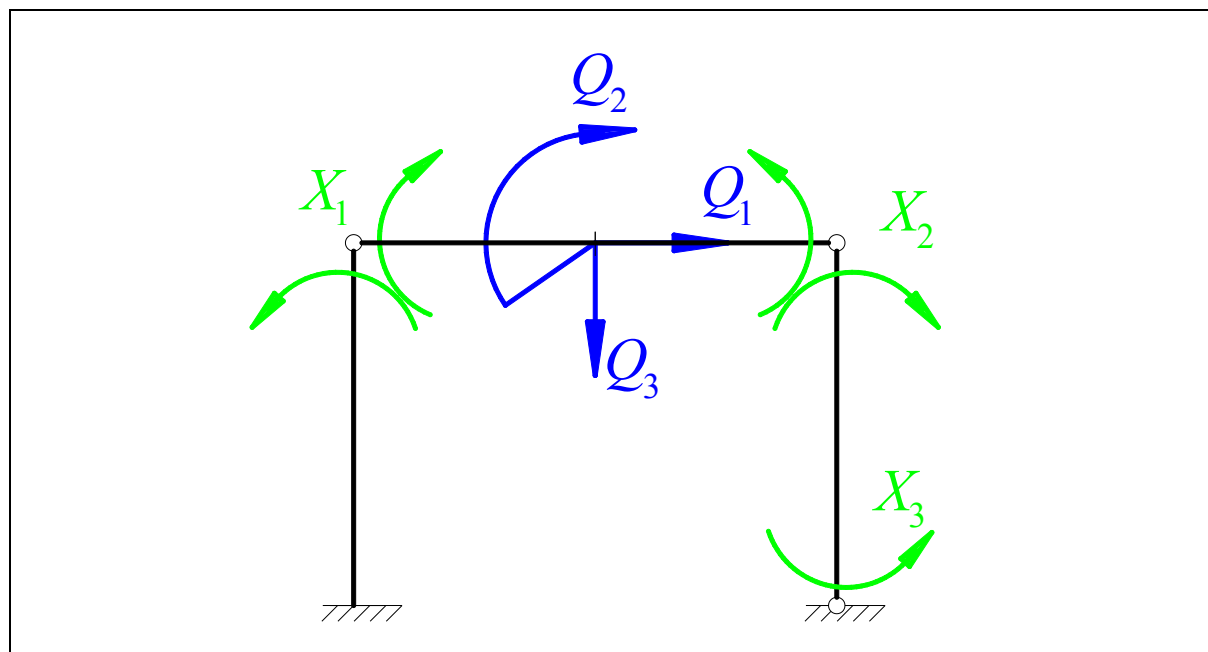
Flexibility matrix as the inverse of stiffness matrix

$$\mathbf{D} = \mathbf{K}^{-1} = \frac{\text{adj}\mathbf{K}}{\det\mathbf{K}}$$

$$\mathbf{D} = \begin{bmatrix} 0.438886 \frac{\text{m}}{\text{MN}} & -0.0272177 \frac{1}{\text{MN}} & 0 \\ -0.0272177 \frac{1}{\text{MN}} & 0.0268817 \frac{1}{\text{MNm}} & 0 \\ 0 & 0 & 0.13 \frac{\text{m}}{\text{MN}} \end{bmatrix}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Force Method – Flexibility matrix



$$\ddot{\mathbf{D}} = \begin{bmatrix} 2.53125 \frac{\text{m}}{\text{MN}} & 0 & 0 & -0.84375 \frac{1}{\text{MN}} & 0.5625 \frac{1}{\text{MN}} & -0.5625 \frac{1}{\text{MN}} \\ 0 & \frac{1}{36} \frac{1}{\text{MNm}} & 0 & -\frac{1}{72} \frac{1}{\text{MNm}} & \frac{1}{72} \frac{1}{\text{MNm}} & 0 \\ 0 & 0 & \frac{1}{4} \frac{\text{m}}{\text{MN}} & \frac{1}{8} \frac{1}{\text{MN}} & \frac{1}{8} \frac{1}{\text{MN}} & 0 \\ -0.84375 \frac{1}{\text{MN}} & -\frac{1}{72} \frac{1}{\text{MNm}} & \frac{1}{8} \frac{1}{\text{MN}} & 0.486111 \frac{1}{\text{MNm}} & -0.131944 \frac{1}{\text{MNm}} & 0.1875 \frac{1}{\text{MNm}} \\ 0.5625 \frac{1}{\text{MN}} & \frac{1}{72} \frac{1}{\text{MNm}} & \frac{1}{8} \frac{1}{\text{MN}} & -0.131944 \frac{1}{\text{MNm}} & 0.361111 \frac{1}{\text{MNm}} & -0.0625 \frac{1}{\text{MNm}} \\ -0.5625 \frac{1}{\text{MN}} & 0 & 0 & 0.1875 \frac{1}{\text{MNm}} & -0.0625 \frac{1}{\text{MNm}} & \frac{1}{4} \frac{1}{\text{MNm}} \end{bmatrix}$$

$$\mathbf{D}_{\text{QQ}} = \begin{bmatrix} 2.53125 \frac{\text{m}}{\text{MN}} & 0 & 0 \\ 0 & \frac{1}{36} \frac{1}{\text{MNm}} & 0 \\ 0 & 0 & \frac{1}{4} \frac{\text{m}}{\text{MN}} \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

$$\mathbf{D}_{QX} = \begin{bmatrix} -0.84375 \frac{1}{\text{MN}} & 0.5625 \frac{1}{\text{MN}} & -0.5625 \frac{1}{\text{MN}} \\ -\frac{1}{72} \frac{1}{\text{MNm}} & \frac{1}{72} \frac{1}{\text{MNm}} & 0 \\ \frac{1}{8} \frac{1}{\text{MN}} & \frac{1}{8} \frac{1}{\text{MN}} & 0 \end{bmatrix}$$

$$\mathbf{D}_{XQ} = \begin{bmatrix} -0.84375 \frac{1}{\text{MN}} & -\frac{1}{72} \frac{1}{\text{MNm}} & \frac{1}{8} \frac{1}{\text{MN}} \\ 0.5625 \frac{1}{\text{MN}} & \frac{1}{72} \frac{1}{\text{MNm}} & \frac{1}{8} \frac{1}{\text{MN}} \\ -0.5625 \frac{1}{\text{MN}} & 0 & 0 \end{bmatrix}$$

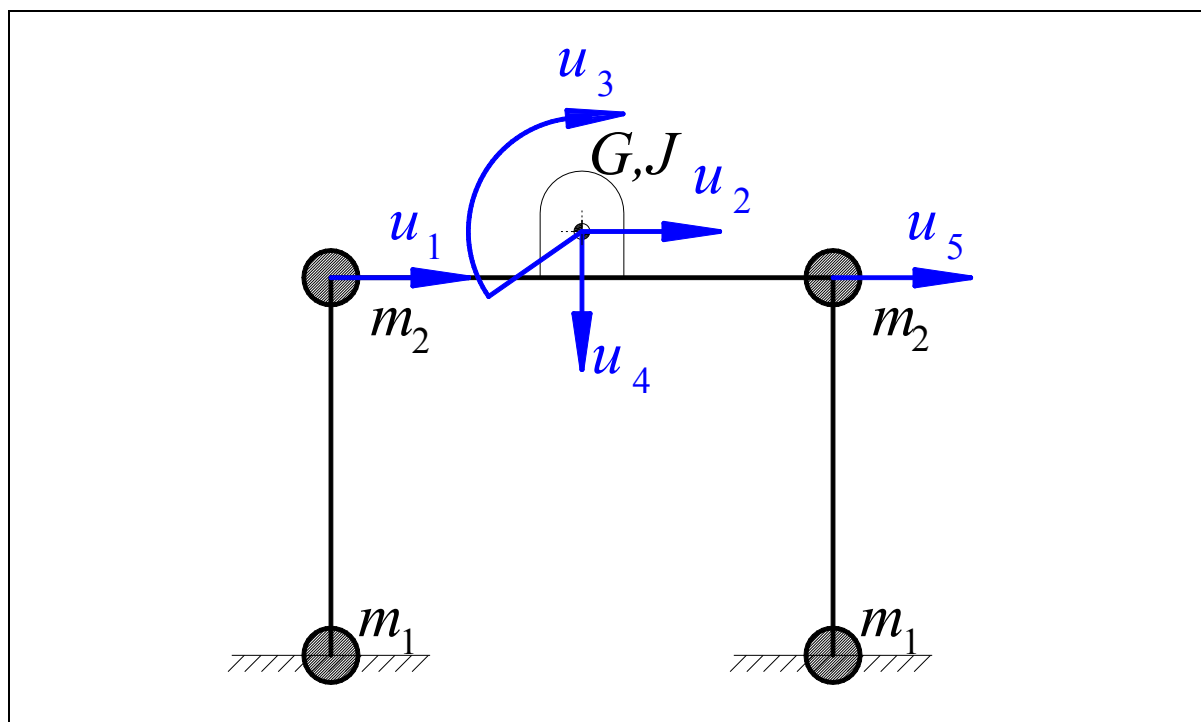
$$\mathbf{D}_{XX} = \begin{bmatrix} 0.486111 \frac{1}{\text{MNm}} & -0.131944 \frac{1}{\text{MNm}} & 0.1875 \frac{1}{\text{MNm}} \\ -0.131944 \frac{1}{\text{MNm}} & 0.361111 \frac{1}{\text{MNm}} & -0.0625 \frac{1}{\text{MNm}} \\ 0.1875 \frac{1}{\text{MNm}} & -0.0625 \frac{1}{\text{MNm}} & \frac{1}{4} \frac{1}{\text{MNm}} \end{bmatrix}$$

$$\mathbf{D} = \mathbf{D}_{QQ} - \mathbf{D}_{QX} \cdot \mathbf{D}_{XX}^{-1} \cdot \mathbf{D}_{XQ}$$

$$\mathbf{D} = \begin{bmatrix} 0.438886 \frac{\text{m}}{\text{MN}} & -0.0272177 \frac{1}{\text{MN}} & 0 \\ -0.0272177 \frac{1}{\text{MN}} & 0.0268817 \frac{1}{\text{MNm}} & 0 \\ 0 & 0 & 0.13 \frac{\text{m}}{\text{MN}} \end{bmatrix}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Inertia matrix



$$u_1 = 1 \cdot q_1 + 0 \cdot q_2 + 0 \cdot q_3$$

$$u_2 = 1 \cdot q_1 + h_0 \cdot q_2 + 0 \cdot q_3$$

$$u_3 = 0 \cdot q_1 + 1 \cdot q_2 + 0 \cdot q_3$$

$$u_4 = 0 \cdot q_1 + 0 \cdot q_2 + 1 \cdot q_3$$

$$u_5 = 1 \cdot q_1 + 0 \cdot q_2 + 0 \cdot q_3$$

$$\mathbf{A}_m = \begin{bmatrix} 1 & 0 & 0 \\ 1 & h_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.25m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$J = \rho \cdot (I_x + I_y) = \frac{G}{h_0 \cdot 2h_0} \left[\frac{h_0 \cdot (2h_0)^2}{12} + \frac{2h_0 \cdot h_0^2}{12} \right] =$$

$$= \frac{500\text{kg}}{0.25\text{m} \cdot 2 \cdot 0.25\text{m}} \left[\frac{0.25\text{m} \cdot (2 \cdot 0.25\text{m})^2}{12} + \frac{2 \cdot 0.25\text{m} \cdot (0.25\text{m})^2}{12} \right] = 13.02 \text{ kgm}^2$$

$$\{\mathbf{m}\} = \text{diag}(m_2 \quad G \quad J \quad G \quad m_2) = \text{diag}(255\text{kg} \quad 500\text{kg} \quad 13.02\text{kgm}^2 \quad 500\text{kg} \quad 255\text{kg}) =$$

$$= \begin{bmatrix} 255\text{kg} & 0 & 0 & 0 & 0 \\ 0 & 500\text{kg} & 0 & 0 & 0 \\ 0 & 0 & 13.02\text{kgm}^2 & 0 & 0 \\ 0 & 0 & 0 & 500\text{kg} & 0 \\ 0 & 0 & 0 & 0 & 255\text{kg} \end{bmatrix}$$

Inertia matrix

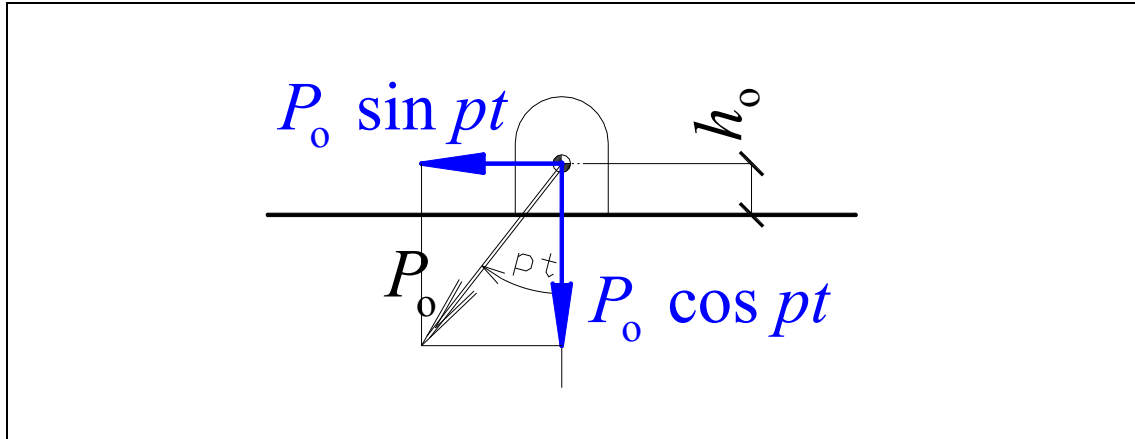
$$\mathbf{B} = \mathbf{A}_m^T \cdot \{\mathbf{m}\} \cdot \mathbf{A}_m =$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0.25\text{m} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 255\text{kg} & 0 & 0 & 0 & 0 \\ 0 & 500\text{kg} & 0 & 0 & 0 \\ 0 & 0 & 500\text{kg} & 0 & 0 \\ 0 & 0 & 0 & 312.5\text{kgm}^2 & 0 \\ 0 & 0 & 0 & 0 & 255\text{kg} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0.25\text{m} & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1010\text{kg} & 125\text{kgm} & 0 \\ 125\text{kgm} & 44.27\text{kgm}^2 & 0 \\ 0 & 0 & 500\text{kg} \end{bmatrix}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Excitation force



An excitation force occurs as a result of the rotation of mass $G_w=200\text{kg}$, which is a part of mass $G=500\text{kg}$ on the eccentricity $e=0.002\text{m}$ with the frequency $n = 400\text{cycle/min}$.

$$f = \frac{n}{60} = \frac{400}{60} \text{ Hz}$$

$$p = 2\pi f = 2\pi \frac{n}{60} = 41.89 \frac{\text{rad}}{\text{s}}$$

$$P_o = G_w \cdot e \cdot p^2 = 200\text{kg} \cdot 0.002\text{m} \cdot 41.89^2 \frac{\text{rad}^2}{\text{s}^2} = 701.89 \text{ N} = 0.70189 \text{ kN}$$

$$\mathbf{F} = \mathbf{u}_s \cdot P_o \sin pt + \mathbf{u}_c \cdot P_o \cos pt =$$

$$= \mathbf{F}_s \cdot \sin pt + \mathbf{F}_c \cdot \cos pt$$

$$\mathbf{u}_s = \begin{bmatrix} -1 \\ -h_0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -0.25\text{m} \\ 0 \end{bmatrix} \quad \mathbf{u}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{F}_s = \begin{bmatrix} -0.70189 \text{ kN} \\ -0.17547 \text{ kNm} \\ 0 \end{bmatrix} \quad \mathbf{F}_c = \begin{bmatrix} 0 \\ 0 \\ 0.70189 \text{ kN} \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Eigenproblem

Equation of motion:

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

For harmonic motion the formula is valid

$$\ddot{\mathbf{q}} = -\omega^2 \mathbf{q}$$

Substituting this expression into equation of motion gives

$$(\mathbf{K} - \omega^2 \mathbf{B})\mathbf{q} = \mathbf{0}$$

$$(\mathbf{K} - \lambda \mathbf{B})\mathbf{q} = \mathbf{0}$$

where

$$\lambda = \omega^2$$

Eigenvalues

The solution is non-trivial if

$$\det(\mathbf{K} - \lambda \mathbf{B}) = 0$$

$$\mathbf{K} - \lambda \mathbf{B} =$$

$$= \begin{bmatrix} 2.43115 \cdot 10^6 \frac{\text{N}}{\text{m}} - \lambda \cdot 1010 \text{kg} & 2.46154 \cdot 10^6 \text{N} - \lambda \cdot 125 \text{kg} & 0 \\ 2.46154 \cdot 10^6 \text{N} - \lambda \cdot 125 \text{kg} & 3.96923 \cdot 10^6 \text{Nm} - \lambda \cdot 44.27 \text{kgm}^2 & 0 \\ 0 & 0 & 7.69231 \cdot 10^6 \frac{\text{N}}{\text{m}} - \lambda \cdot 500 \text{kg} \end{bmatrix}$$

$$\det(\mathbf{K} - \lambda \mathbf{B}) = 6.95683 \cdot 10^{20} - 3.49692 \cdot 10^{17} \lambda + 2.00145 \cdot 10^{13} \lambda^2 - 1.45438 \cdot 10^7 \lambda^3 = 0$$

Using the *Mathematica* system one can achieve eigenvalues and natural frequencies

$$\lambda_1 = 2288.73 \cdot \frac{1}{\text{s}^2}$$

$$\omega_1 = 47.84 \cdot \frac{\text{rad}}{\text{s}}$$

$$f_1 = 7.614 \cdot \text{Hz}$$

$$\lambda_2 = 15384.6 \cdot \frac{1}{\text{s}^2}$$

$$\omega_2 = 124.0 \cdot \frac{\text{rad}}{\text{s}}$$

$$f_2 = 19.74 \cdot \text{Hz}$$

$$\lambda_3 = 1358470 \cdot \frac{1}{\text{s}^2}$$

$$\omega_3 = 1166 \cdot \frac{\text{rad}}{\text{s}}$$

$$f_3 = 185.5 \cdot \text{Hz}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Calculation of Modal Matrix

Eigenforms

Each non-zero column of the adjoint matrix $\text{adj}\mathbf{A}_i$ is an eigenvector $\mathbf{w}_i$

$$\text{adj}\mathbf{A}_i = \text{adj} [\mathbf{K} - \lambda_i \cdot \mathbf{B}]$$

$$\mathbf{A}_1 = \mathbf{K} - \lambda_1 \cdot \mathbf{B} = \begin{bmatrix} 0.119537 \cdot 10^6 \frac{\text{N}}{\text{m}} & 2.17545 \cdot 10^6 \text{ N} & 0 \\ 2.17545 \cdot 10^6 \text{ N} & 3.9591 \cdot 10^7 \text{ Nm} & 0 \\ 0 & 0 & 6.54795 \cdot 10^6 \frac{\text{N}}{\text{m}} \end{bmatrix}$$

$$\text{adj}\mathbf{A}_1 = \begin{bmatrix} 2.5924 \cdot 10^{14} \text{ N}^2 & -1.42447 \cdot 10^{13} \frac{\text{N}^2}{\text{m}} & 0 \\ -1.42447 \cdot 10^{13} \frac{\text{N}^2}{\text{m}} & 7.82719 \cdot 10^{19} \frac{\text{N}^2}{\text{m}^2} & 0 \\ 0 & 0 & -0.038 \text{ N}^2 \approx 0 \end{bmatrix}$$

After normalization of the columns of $\text{adj}\mathbf{A}_1$ matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ -0.0549481 \frac{1}{\text{m}} & 0.0549481 \frac{1}{\text{m}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{w}_1 = \begin{bmatrix} 1 \\ -0.0549481 \frac{1}{\text{m}} \\ 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{K} - \lambda_2 \cdot \mathbf{B} = \begin{bmatrix} -1.31073 \cdot 10^7 \frac{\text{N}}{\text{m}} & 0.538462 \cdot 10^6 \text{ N} & 0 \\ 0.538462 \cdot 10^6 \text{ N} & 3.90112 \cdot 10^7 \text{ Nm} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{adj}\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5.11622 \cdot 10^{14} \text{ N}^2 \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

After normalization of the columns of $\text{adj}\mathbf{A}_1$ matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{K} - \lambda_3 \cdot \mathbf{B} = \begin{bmatrix} -1.36963 \cdot 10^9 \frac{\text{N}}{\text{m}} & -1.67348 \cdot 10^8 \text{N} & 0 \\ -1.67348 \cdot 10^8 \text{N} & -2.04474 \cdot 10^7 \text{Nm} & 0 \\ 0 & 0 & 6.71545 \cdot 10^8 \frac{\text{N}}{\text{m}} \end{bmatrix}$$

$$\text{adj}\mathbf{A}_1 = \begin{bmatrix} 1.37313 \cdot 10^{16} \text{N}^2 & -1.12382 \cdot 10^{17} \frac{\text{N}^2}{\text{m}} & 0 \\ -1.12382 \cdot 10^{17} \frac{\text{N}^2}{\text{m}} & 9.19767 \cdot 10^{17} \frac{\text{N}^2}{\text{m}^2} & 0 \\ 0 & 0 & -4\text{N}^2 \approx 0 \end{bmatrix}$$

After normalization of the columns of $\text{adj}\mathbf{A}_1$ matrix

$$\begin{bmatrix} 0.122185 & -0.122185 & 0 \\ -1 \frac{1}{\text{m}} & 1 \frac{1}{\text{m}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{w}_1 = \begin{bmatrix} -0.122185 \\ 1 \frac{1}{\text{m}} \\ 0 \end{bmatrix}$$

Modal matrix

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Static displacement of modal forms

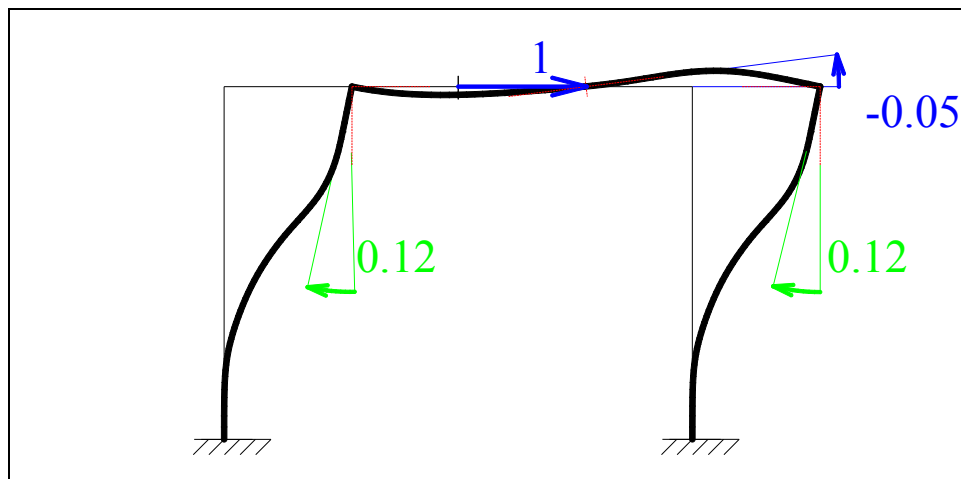
$$\begin{aligned}
 \mathbf{W}_x &= \mathbf{K}_{xx}^{-1} \cdot \mathbf{K}_{xq} \cdot \mathbf{W} = \\
 &= \begin{bmatrix} -3.556 \text{ MN} & -3.556 \text{ MN} \\ 12 \text{ MNm} & 12 \text{ MNm} \\ -12 \text{ MN} & 12 \text{ MN} \end{bmatrix}^{-1} \cdot \\
 &\cdot \begin{bmatrix} 34,667 \text{ MNm} & 0 \\ 0 & 34,667 \text{ MNm} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \end{bmatrix} = \\
 &= \begin{bmatrix} 0.121585 \frac{1}{\text{m}} & -0.346154 \frac{1}{\text{m}} & -0.358686 \frac{1}{\text{m}} \\ 0.121585 \frac{1}{\text{m}} & 0.346154 \frac{1}{\text{m}} & -0.358686 \frac{1}{\text{m}} \end{bmatrix}
 \end{aligned}$$

Modal matrix in an expanded base of coordinates

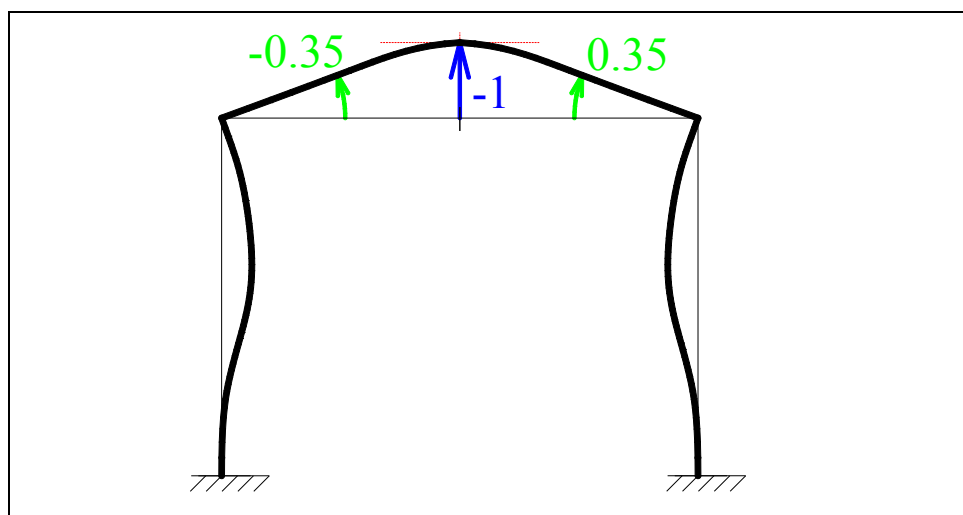
$$\tilde{\mathbf{W}} = \begin{bmatrix} \mathbf{W} \\ \mathbf{W}_x \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \\ 0.121585 \frac{1}{\text{m}} & -0.346154 \frac{1}{\text{m}} & -0.358686 \frac{1}{\text{m}} \\ 0.121585 \frac{1}{\text{m}} & 0.346154 \frac{1}{\text{m}} & -0.358686 \frac{1}{\text{m}} \end{bmatrix}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

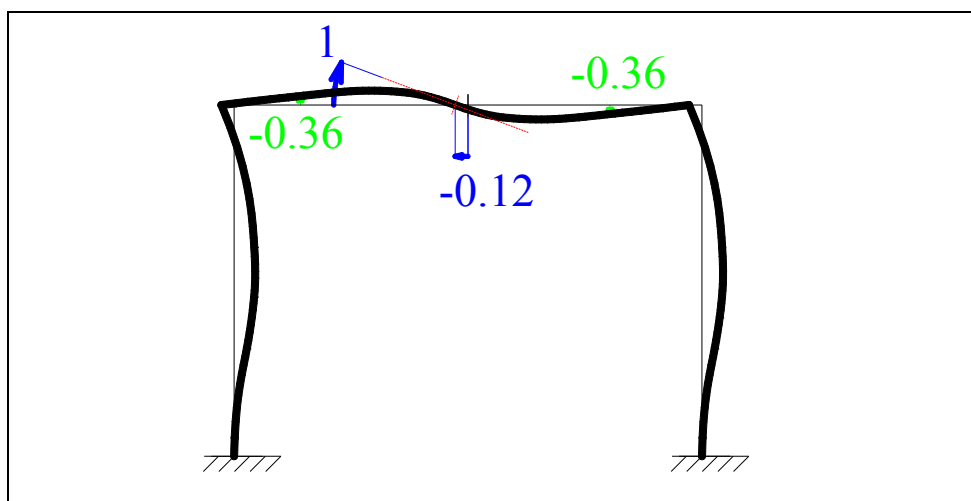
First eigenform



Second eigenform



Third eigenform





THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Harmonic forced vibration

Voigt-Kelvin model of damping – $\gamma = 2\alpha = 0.2$

$$\mathbf{C} = \frac{\gamma}{p} \mathbf{K}$$

Modal Transformation Method

Matrix equation of motion in a principle coordinates base

$$\{\mathbf{m}_0\}\ddot{\mathbf{r}} + \{\mathbf{c}_0\}\dot{\mathbf{r}} + \{\mathbf{k}_0\}\mathbf{r} = \mathbf{R}_S \cdot \sin pt + \mathbf{R}_C \cdot \cos pt$$

Stiffness matrix in a principle coordinates base

$$\{\mathbf{k}_0\} = \mathbf{W}^T \mathbf{K} \mathbf{W} =$$

$$= \begin{bmatrix} 1 & -0.0549481 & \frac{1}{m} & 0 \\ 0 & 0 & 0 & -1 \\ -0.122185 & 1 & \frac{1}{m} & 0 \end{bmatrix} \cdot \begin{bmatrix} 2.43115 \frac{\text{MN}}{m} & 2.46154 \text{ MN} & 0 \\ 2.46154 \text{ MN} & 39.6923 \text{ MNm} & 0 \\ 0 & 0 & 7.69231 \frac{\text{MN}}{m} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 & \frac{1}{m} & 0 \\ 0 & -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2.28048 \frac{\text{MN}}{m} & 0 & 0 \\ 0 & 7.69231 \frac{\text{MN}}{m} & 0 \\ 0 & 0 & 39.1271 \frac{\text{MN}}{m} \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Mass matrix in a principle coordinates base

$$\begin{aligned}\{\mathbf{m}_0\} &= \mathbf{W}^T \mathbf{B} \mathbf{W} = \\ &= \begin{bmatrix} 1 & -0.0549481 \frac{1}{\text{m}} & 0 \\ 0 & 0 & -1 \\ -0.122185 & 1 \frac{1}{\text{m}} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1010 \text{ kg} & 125 \text{ kgm} & 0 \\ 125 \text{ kgm} & 44.27 \text{ kgm}^2 & 0 \\ 0 & 0 & 500 \text{ kg} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} 996.397 \text{ kg} & 0 & 0 \\ 0 & 500 \text{ kg} & 0 \\ 0 & 0 & 28.8022 \text{ kg} \end{bmatrix}\end{aligned}$$

Damping matrix in a principle coordinates base

$$\begin{aligned}\{\mathbf{c}_0\} &= \frac{\gamma}{p} \{\mathbf{k}_0\} = \\ &= \frac{0.2}{41,89 \frac{\text{rad}}{\text{s}}} \begin{bmatrix} 2.28048 \frac{\text{MN}}{\text{m}} & 0 & 0 \\ 0 & 7.69231 \frac{\text{MN}}{\text{m}} & 0 \\ 0 & 0 & 39.1271 \frac{\text{MN}}{\text{m}} \end{bmatrix} = \\ &= \begin{bmatrix} 10887.8 \frac{\text{kg}}{\text{s}} & 0 & 0 \\ 0 & 36726.2 \frac{\text{kg}}{\text{s}} & 0 \\ 0 & 0 & 186808 \frac{\text{kg}}{\text{s}} \end{bmatrix}\end{aligned}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Excitation force vector in a principle coordinates base

$$\mathbf{R}(t) = \mathbf{W}^T \mathbf{F} = \mathbf{W}^T (\mathbf{F}_S \cdot \sin pt + \mathbf{F}_C \cdot \cos pt) =$$

$$\begin{bmatrix} 1 & -0.0549481 \frac{1}{\text{m}} & 0 \\ 0 & 0 & -1 \\ -0.122185 & 1 \frac{1}{\text{m}} & 0 \end{bmatrix} \cdot$$

$$\cdot \left(\begin{bmatrix} -0.70189 \text{ kN} \\ -0.17547 \text{ kNm} \\ 0 \end{bmatrix} \cdot \sin pt + \begin{bmatrix} 0 \\ 0 \\ 0.70189 \text{ kN} \end{bmatrix} \cdot \cos pt \right) =$$

$$= \begin{bmatrix} -0.692248 \text{ kN} \\ 0 \\ -0.0897096 \text{ kN} \end{bmatrix} \cdot \sin pt + \begin{bmatrix} 0 \\ -0.70189 \text{ kN} \\ 0 \end{bmatrix} \cdot \cos pt$$

$$\mathbf{R}_S = \begin{bmatrix} -0.692248 \text{ kN} \\ 0 \\ -0.0897096 \text{ kN} \end{bmatrix} = \begin{bmatrix} -692.248 \text{ N} \\ 0 \\ -89.7096 \text{ N} \end{bmatrix}$$

$$\mathbf{R}_C = \begin{bmatrix} 0 \\ -0.70189 \text{ kN} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -701.89 \text{ N} \\ 0 \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Solution of the equation of motion

Undaped system

$$\gamma = 0$$

$$h_{1i} = \frac{1}{k_{0i}} \frac{1}{1 - \eta_i^2}$$

$$h_{2i} = 0$$

$$\eta_i = \frac{p}{\omega_i}$$

$$\{\mathbf{h}_1\} = \begin{bmatrix} 1.087959 \cdot 10^{-6} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 1.46737 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 2.55908 \cdot 10^{-8} \frac{\text{m}}{\text{N}} \end{bmatrix}$$

Solution in a principle coordinates base

$$\mathbf{r}_s = \{\mathbf{h}_1\} \cdot \mathbf{R}_s =$$

$$= \begin{bmatrix} 1.087959 \cdot 10^{-6} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 1.46737 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 2.55908 \cdot 10^{-8} \frac{\text{m}}{\text{N}} \end{bmatrix} \cdot \begin{bmatrix} -692.248 \text{ N} \\ 0 \\ -89.7096 \text{ N} \end{bmatrix} =$$

$$= \begin{bmatrix} -1.30114 \cdot 10^{-3} \text{ m} \\ 0 \\ -2.29574 \cdot 10^{-6} \text{ m} \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$\begin{aligned} \mathbf{r}_C &= \{\mathbf{h}_1\} \cdot \mathbf{R}_C = \\ &= \begin{bmatrix} 1.087959 \cdot 10^{-6} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 1.46737 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 2.55908 \cdot 10^{-8} \frac{\text{m}}{\text{N}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -701.89 \text{ N} \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} 0 \\ -0.102993 \cdot 10^{-3} \text{ m} \\ 0 \end{bmatrix} \end{aligned}$$

Solution in a generalized coordinates base – $\mathbf{q} = \mathbf{W}\mathbf{r}$

$$\begin{aligned} \mathbf{q}_S &= \mathbf{W} \cdot \mathbf{r}_S = \\ &= \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1.30114 \cdot 10^{-3} \text{ m} \\ 0 \\ -2.29574 \cdot 10^{-6} \text{ m} \end{bmatrix} = \\ &= \begin{bmatrix} -1.30086 \cdot 10^{-3} \text{ m} \\ 6.91996 \cdot 10^{-5} \text{ rad} \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{q}_C &= \mathbf{W} \cdot \mathbf{r}_S = \\ &= \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -0.102993 \cdot 10^{-3} \text{ m} \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} 0 \\ 0 \\ 1.02993 \cdot 10^{-4} \text{ m} \end{bmatrix} \end{aligned}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Damped system

$$h_{1i} = \frac{1}{k_{0i}} \frac{1 - \eta_i^2}{(1 - \eta_i^2)^2 + (\gamma_i \eta_i)^2} \quad h_{2i} = \frac{1}{k_{0i}} \frac{\gamma_i \eta_i}{(1 - \eta_i^2)^2 + (\gamma_i \eta_i)^2}$$

$$\gamma_i = \frac{\gamma}{\eta_i} \quad \eta_i = \frac{p}{\omega_i}$$

$$\{\mathbf{h}_1\} = \begin{bmatrix} 1.08339 \cdot 10^{-6} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 1.39621 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 2.46041 \cdot 10^{-8} \frac{\text{m}}{\text{N}} \end{bmatrix}$$

$$\{\mathbf{h}_2\} = \begin{bmatrix} 9.28761 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 3.15194 \cdot 10^{-8} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 4.92718 \cdot 10^{-9} \frac{\text{m}}{\text{N}} \end{bmatrix}$$

Solution in a principle coordinates base

$$\mathbf{r}_S = \{\mathbf{h}_1\} \cdot \mathbf{R}_S + \{\mathbf{h}_2\} \cdot \mathbf{R}_C =$$

$$= \begin{bmatrix} 1.08339 \cdot 10^{-6} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 1.39621 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 2.46041 \cdot 10^{-8} \frac{\text{m}}{\text{N}} \end{bmatrix} \cdot \begin{bmatrix} -692.248 \text{ N} \\ 0 \\ -89.7096 \text{ N} \end{bmatrix} +$$

$$+ \begin{bmatrix} 9.28761 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 3.15194 \cdot 10^{-8} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 4.92718 \cdot 10^{-9} \frac{\text{m}}{\text{N}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -701.89 \text{ N} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.749974 \cdot 10^{-3} \text{ m} \\ -0.221231 \cdot 10^{-4} \text{ m} \\ -2.20722 \cdot 10^{-6} \text{ m} \end{bmatrix}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$\begin{aligned}
 \mathbf{r}_C &= \{\mathbf{h}_1\} \cdot \mathbf{R}_C - \{\mathbf{h}_2\} \cdot \mathbf{R}_S = \\
 &= \begin{bmatrix} 1.08339 \cdot 10^{-6} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 1.39621 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 2.46041 \cdot 10^{-8} \frac{\text{m}}{\text{N}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -701.89 \text{ N} \\ 0 \end{bmatrix} + \\
 &- \begin{bmatrix} 9.28761 \cdot 10^{-7} \frac{\text{m}}{\text{N}} & 0 & 0 \\ 0 & 3.15194 \cdot 10^{-8} \frac{\text{m}}{\text{N}} & 0 \\ 0 & 0 & 4.92718 \cdot 10^{-9} \frac{\text{m}}{\text{N}} \end{bmatrix} \cdot \begin{bmatrix} -692.248 \text{ N} \\ 0 \\ -89.7096 \text{ N} \end{bmatrix} = \\
 &= \begin{bmatrix} 6.42933 \cdot 10^{-4} \text{ m} \\ -0.979988 \cdot 10^{-4} \text{ m} \\ 4.42015 \cdot 10^{-7} \text{ m} \end{bmatrix}
 \end{aligned}$$

Solution in a generalized coordinates base – $\mathbf{q} = \mathbf{W}\mathbf{r}$

$$\begin{aligned}
 \mathbf{q}_S &= \mathbf{W} \cdot \mathbf{r}_S = \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -0.749974 \cdot 10^{-3} \text{ m} \\ -0.221231 \cdot 10^{-4} \text{ m} \\ -2.20722 \cdot 10^{-6} \text{ m} \end{bmatrix} = \\
 &= \begin{bmatrix} -0.749704 \cdot 10^{-3} \text{ m} \\ 3.90024 \cdot 10^{-5} \text{ rad} \\ 2.21231 \cdot 10^{-5} \text{ m} \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{q}_C &= \mathbf{W} \cdot \mathbf{r}_C = \begin{bmatrix} 1 & 0 & -0.122185 \\ -0.0549481 \frac{1}{\text{m}} & 0 & 1 \frac{1}{\text{m}} \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6.42933 \cdot 10^{-4} \text{ m} \\ -0.979988 \cdot 10^{-4} \text{ m} \\ 4.42015 \cdot 10^{-7} \text{ m} \end{bmatrix} = \\
 &= \begin{bmatrix} 6.42879 \cdot 10^{-4} \text{ m} \\ -3.48859 \cdot 10^{-5} \text{ rad} \\ 0.979988 \cdot 10^{-4} \text{ m} \end{bmatrix}
 \end{aligned}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Direct Method

Undaped system

$$\begin{aligned}
 \mathbf{q}_S &= (\mathbf{K} - p^2 \cdot \mathbf{B}) \cdot \mathbf{F}_S = \begin{bmatrix} 2.43115 \frac{\text{MN}}{\text{m}} & 2.46154 \text{MN} & 0 \\ 2.46154 \text{MN} & 39.6923 \text{MNm} & 0 \\ 0 & 0 & 7.69231 \frac{\text{MN}}{\text{m}} \end{bmatrix} + \\
 &- \left(41.89 \frac{\text{rad}}{\text{s}} \right)^2 \cdot \begin{bmatrix} 1010 \text{kg} & 125 \text{kgm} & 0 \\ 125 \text{kgm} & 44.27 \text{kgm}^2 & 0 \\ 0 & 0 & 500 \text{kg} \end{bmatrix} \cdot \begin{bmatrix} -0.70189 \text{kN} \\ -0.17547 \text{kNm} \\ 0 \end{bmatrix} = \\
 &= \begin{bmatrix} -1.30034 \cdot 10^{-3} \text{m} \\ 6.91711 \cdot 10^{-5} \text{rad} \\ 0 \end{bmatrix} \\
 \mathbf{x}_S &= \mathbf{K}_{xx}^{-1} \cdot \mathbf{K}_{xq} \cdot \mathbf{q}_S = \\
 &= \begin{bmatrix} -3.556 \text{MN} & -3.556 \text{MN} \\ 12 \text{MNm} & 12 \text{MNm} \\ -12 \text{MN} & 12 \text{MN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 34,667 \text{MNm} & 0 \\ 0 & 34,667 \text{MNm} \end{bmatrix} \cdot \begin{bmatrix} -1.30034 \cdot 10^{-3} \text{m} \\ 6.91711 \cdot 10^{-5} \text{rad} \\ 0 \end{bmatrix} = \\
 &= \begin{bmatrix} -1.573 \cdot 10^{-4} \text{rad} \\ -1.573 \cdot 10^{-4} \text{rad} \end{bmatrix} \\
 \hat{\mathbf{q}}_S &= \begin{bmatrix} \mathbf{q}_S \\ \mathbf{x}_S \end{bmatrix} = \begin{bmatrix} q_{1S} \\ q_{2S} \\ q_{3S} \\ x_{1S} \\ x_{2S} \end{bmatrix} = \begin{bmatrix} -1.300 \cdot 10^{-3} \text{m} \\ 6.917 \cdot 10^{-5} \text{rad} \\ 0 \\ -1.573 \cdot 10^{-4} \text{rad} \\ -1.573 \cdot 10^{-4} \text{rad} \end{bmatrix}
 \end{aligned}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$\mathbf{q}_C = (\mathbf{K} - p^2 \cdot \mathbf{B}) \cdot \mathbf{F}_C = \begin{bmatrix} 2.43115 \frac{\text{MN}}{\text{m}} & 2.46154 \text{ MN} & 0 \\ 2.46154 \text{ MN} & 39.6923 \text{ MNm} & 0 \\ 0 & 0 & 7.69231 \frac{\text{MN}}{\text{m}} \end{bmatrix} +$$

$$-\left(41.89 \frac{\text{rad}}{\text{s}}\right)^2 \cdot \begin{bmatrix} 1010 \text{ kg} & 125 \text{ kgm} & 0 \\ 125 \text{ kgm} & 44.27 \text{ kgm}^2 & 0 \\ 0 & 0 & 500 \text{ kg} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0.70189 \text{ kN} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1.02984 \cdot 10^{-4} \text{ m} \end{bmatrix}$$

$$\mathbf{x}_C = \mathbf{K}_{xx}^{-1} \cdot \mathbf{K}_{xq} \cdot \mathbf{q}_C =$$

$$= \begin{bmatrix} -3.556 \text{ MN} & -3.556 \text{ MN} \\ 12 \text{ MNm} & 12 \text{ MNm} \\ -12 \text{ MN} & 12 \text{ MN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 34,667 \text{ MNm} & 0 \\ 0 & 34,667 \text{ MNm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1.02984 \cdot 10^{-4} \text{ m} \end{bmatrix} =$$

$$= \begin{bmatrix} 3.565 \cdot 10^{-5} \text{ rad} \\ -3.565 \cdot 10^{-5} \text{ rad} \end{bmatrix}$$

$$\hat{\mathbf{q}}_C = \begin{bmatrix} \mathbf{q}_C \\ \mathbf{x}_C \end{bmatrix} = \begin{bmatrix} q_{1C} \\ q_{2C} \\ q_{3C} \\ x_{1C} \\ x_{2C} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.030 \cdot 10^{-4} \text{ m} \\ 3.565 \cdot 10^{-5} \text{ rad} \\ -3.565 \cdot 10^{-5} \text{ rad} \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Dynamic displacements of points

$$\text{am } q_i = \sqrt{q_{iS}^2 + q_{iC}^2}$$

$$\text{am } \hat{\mathbf{q}} = \begin{bmatrix} \text{am } \mathbf{q} \\ \text{am } \mathbf{x} \end{bmatrix} = \begin{bmatrix} \text{am } q_1 \\ \text{am } q_2 \\ \text{am } q_3 \\ \text{am } x_1 \\ \text{am } x_2 \end{bmatrix} = \begin{bmatrix} 1.3 \cdot 10^{-3} \text{ m} \\ 6.9 \cdot 10^{-5} \text{ rad} \\ 1.03 \cdot 10^{-4} \text{ m} \\ 1.61 \cdot 10^{-4} \text{ rad} \\ 1.61 \cdot 10^{-4} \text{ rad} \end{bmatrix}$$

Bending moments

$$M_i = M_i^{q_1=1} \cdot q_1 + M_i^{q_2=1} \cdot q_2 + M_i^{q_3=1} \cdot q_3 + M_i^{x_1=1} \cdot x_1 + M_i^{x_2=1} \cdot x_2$$

Sinusoidal component of dynamic bending moments

$$\mathbf{M}_S = \begin{bmatrix} M_A \\ M_{AC} \\ M_C \\ M_C \\ M_{CD} \\ M_D \\ M_D \\ M_{DE} \\ M_E \\ M_E \\ M_{EB} \\ M_B \end{bmatrix} = \begin{bmatrix} 3.78444 \\ 0.4195 \\ -2.94544 \\ -2.94544 \\ -1.3589 \\ 0.227642 \\ -0.227642 \\ 1.3589 \\ 2.94544 \\ 2.94544 \\ -0.4195 \\ -3.78444 \end{bmatrix} [\text{kNm}]$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Cosinusoidal component of dynamic bending moments

$$\mathbf{M}_c = \begin{bmatrix} M_A \\ M_{AC} \\ M_C \\ M_C \\ M_{CD} \\ M_D \\ M_D \\ M_{DE} \\ M_E \\ M_E \\ M_{EB} \\ M_B \end{bmatrix} = \begin{bmatrix} 0.190125 \\ -0.0950624 \\ -0.380249 \\ -0.380249 \\ 0.21389 \\ 0.80803 \\ 0.80803 \\ 0.21389 \\ -0.380249 \\ -0.380249 \\ -0.0950624 \\ 0.190125 \end{bmatrix} [\text{kNm}]$$

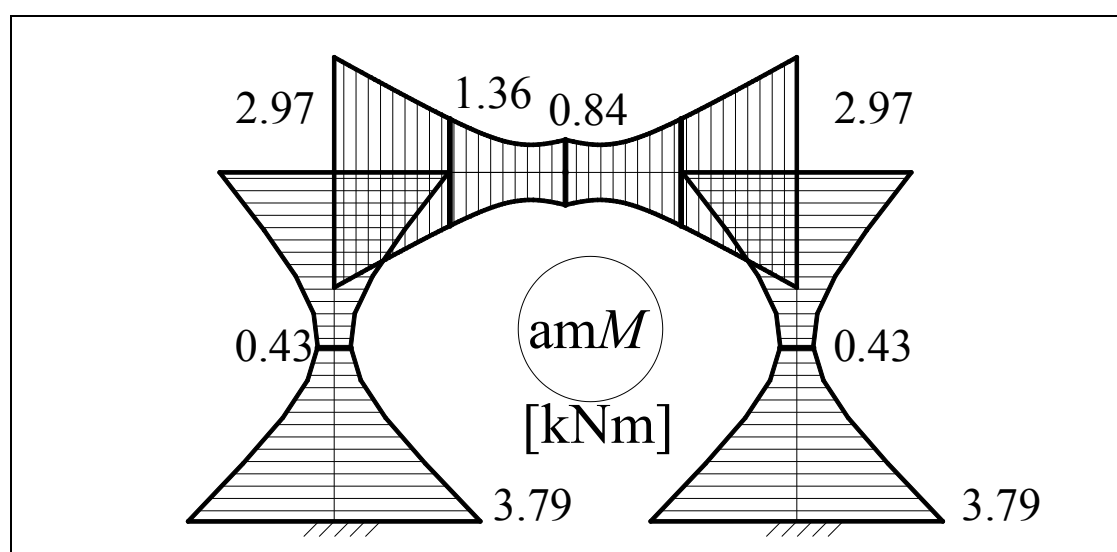
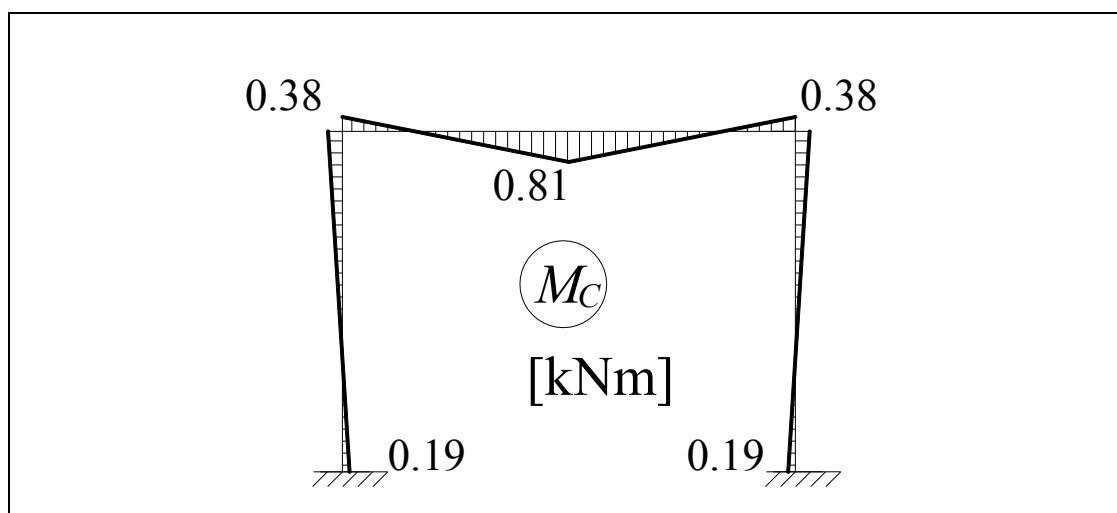
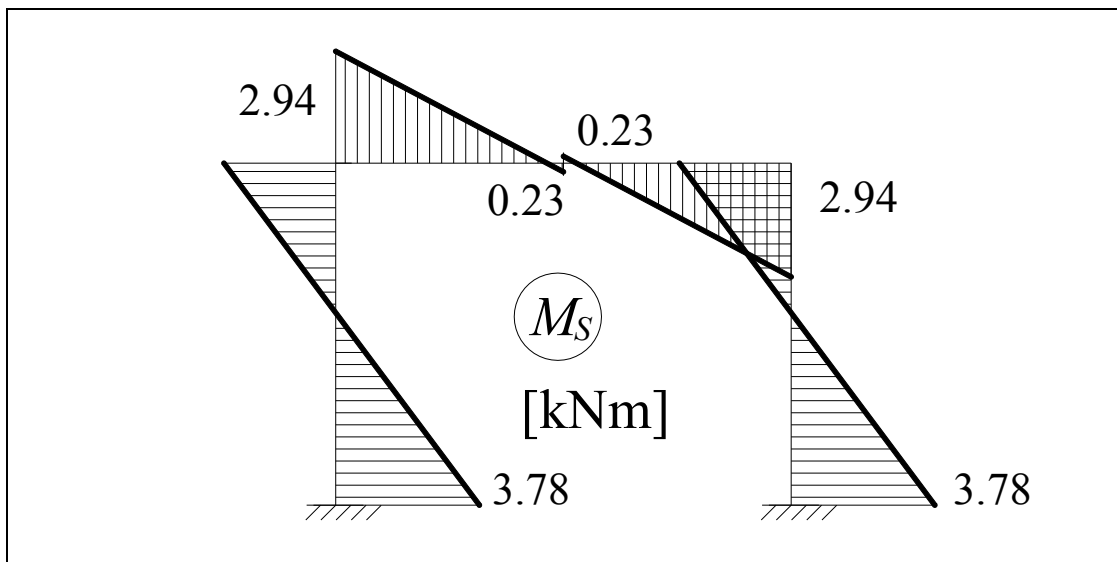
Envelope of total bending moments diagram

$$\text{am } M_i = \sqrt{M_{iS}^2 + M_{iC}^2}$$

$$\text{am } \mathbf{M} = \begin{bmatrix} \text{am } M_A \\ \text{am } M_{AC} \\ \text{am } M_C \\ \text{am } M_C \\ \text{am } M_{CD} \\ \text{am } M_D \\ \text{am } M_D \\ \text{am } M_{DE} \\ \text{am } M_E \\ \text{am } M_E \\ \text{am } M_{EB} \\ \text{am } M_B \end{bmatrix} = \begin{bmatrix} 3.79 \\ 0.43 \\ 2.97 \\ 2.97 \\ 1.36 \\ 0.84 \\ 0.84 \\ 1.36 \\ 2.97 \\ 2.97 \\ 0.43 \\ 3.79 \end{bmatrix} [\text{kNm}]$$

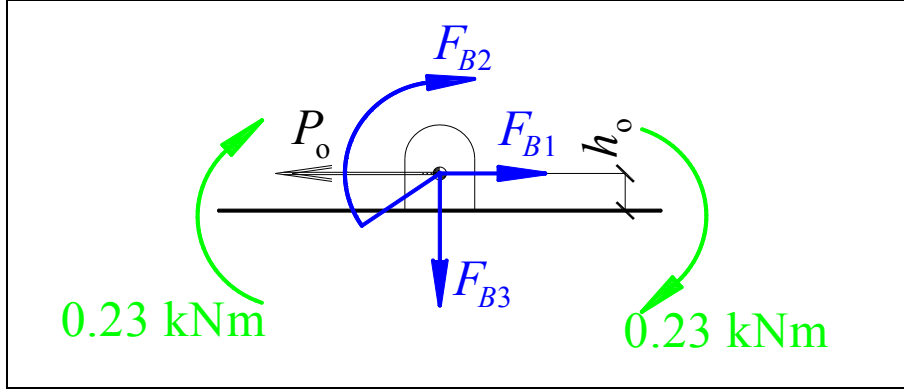
THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Bending moments diagrams



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Checking of the dynamic equilibrium conditions of the node 3



In time point t when

$$pt = 2k\pi + \frac{\pi}{2} \rightarrow \begin{cases} \sin pt = 1 \\ \cos pt = 0 \end{cases} \rightarrow \begin{cases} M = M_s \\ F = F_s \\ q = q_s \end{cases}$$

Inertia forces

$$\begin{aligned} F_{B1} &= -G \cdot (\ddot{q}_{1s} + h_0 \cdot \ddot{q}_{2s}) = -G \cdot (-p^2 \cdot q_{1s} - p^2 \cdot h_0 \cdot q_{2s}) = G \cdot p^2 \cdot (q_{1s} + h_0 \cdot q_{2s}) = \\ &= 500 \text{ kg} \cdot \left(41.89 \frac{\text{rad}}{\text{s}} \right)^2 \cdot (-1.300 \cdot 10^{-3} \text{ m} + 0.25 \text{ m} \cdot 6.917 \cdot 10^{-5} \text{ rad}) = -1125.62 \text{ N} \end{aligned}$$

$$F_{B2} = -J \cdot \ddot{q}_2 = p^2 \cdot J \cdot q_2 = \left(41.89 \frac{\text{rad}}{\text{s}} \right)^2 \cdot 13.02 \text{ kgm}^2 \cdot 6.917 \cdot 10^{-5} \text{ rad} = 1.580 \text{ Nm}$$

$$F_{B3} = -G \cdot \ddot{q}_3 = p^2 \cdot G \cdot q_3 = \left(41.89 \frac{\text{rad}}{\text{s}} \right)^2 \cdot 500 \text{ kg} \cdot 0 \text{ m} = 0 \text{ Nm}$$

Equilibrium conditions – sum of the moments

$$\begin{aligned} \sum M_3 &= 0 \rightarrow 2 \cdot M_{3s} - P_0 \cdot h_0 + F_{B1} \cdot h_0 + F_{B2} + F_{B3} \cdot 0 = 0 \\ 2 \cdot 227.6 \text{ Nm} - 701.89 \text{ N} \cdot 0.25 \text{ m} - 1125.62 \text{ N} \cdot 0.25 \text{ m} + 1.580 \text{ Nm} &= -0.1 \text{ Nm} \approx 0 \end{aligned}$$

In time point t when

$$pt = 2k\pi \rightarrow \begin{cases} \sin pt = 0 \\ \cos pt = 1 \end{cases} \rightarrow \begin{cases} M = M_c \\ F = F_c \\ q = q_c \end{cases} \rightarrow q_{1c} = q_{2c} = 0 \rightarrow F_{B1} = F_{B2} = 0 \rightarrow \sum M_D \equiv 0$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Direct Method

Damped system

$$(\mathbf{K} - p^2 \mathbf{B}) \mathbf{q}_s - p \mathbf{C} \mathbf{q}_c = \mathbf{F}_s$$

$$p \mathbf{C} \mathbf{q}_s + (\mathbf{K} - p^2 \mathbf{B}) \mathbf{q}_c = \mathbf{F}_c$$

Voigt-Kelvin model of damping – $\gamma = 2\alpha = 0.2$

$$\mathbf{C} = \frac{\gamma}{p} \mathbf{K}$$

$$(\mathbf{K} - p^2 \mathbf{B}) \mathbf{q}_s - \gamma \mathbf{K} \mathbf{q}_c = \mathbf{F}_s$$

$$\gamma \mathbf{K} \mathbf{q}_s + (\mathbf{K} - p^2 \mathbf{B}) \mathbf{q}_c = \mathbf{F}_c$$

The set of algebraic equations was solved using the *Mathematica* system

$$\hat{\mathbf{q}}_s = \begin{bmatrix} -0.745 \cdot 10^{-3} \text{ m} \\ 3.900 \cdot 10^{-5} \text{ rad} \\ 2.221 \cdot 10^{-5} \text{ m} \\ -8.273 \cdot 10^{-5} \text{ rad} \\ -4.406 \cdot 10^{-5} \text{ rad} \end{bmatrix} \quad \hat{\mathbf{q}}_c = \begin{bmatrix} 6.426 \cdot 10^{-4} \text{ m} \\ -3.487 \cdot 10^{-5} \text{ rad} \\ 0.9799 \cdot 10^{-4} \text{ m} \\ 1.119 \cdot 10^{-4} \text{ rad} \\ 0.4406 \cdot 10^{-4} \text{ rad} \end{bmatrix}$$

$$\text{am } \hat{\mathbf{q}} = \begin{bmatrix} \text{am } \mathbf{q} \\ \text{am } \mathbf{x} \end{bmatrix} = \begin{bmatrix} \text{am } q_1 \\ \text{am } q_2 \\ \text{am } q_3 \\ \text{am } x_1 \\ \text{am } x_2 \end{bmatrix} = \begin{bmatrix} 0.99 \cdot 10^{-3} \text{ m} \\ 5.23 \cdot 10^{-5} \text{ rad} \\ 1.00 \cdot 10^{-4} \text{ m} \\ 1.39 \cdot 10^{-4} \text{ rad} \\ 1.07 \cdot 10^{-4} \text{ rad} \end{bmatrix}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Bending moments (the elastic forces only)

$$M_{ij} = M_{ij}^{q_1=1} \cdot q_1 + M_{ij}^{q_2=1} \cdot q_2 + M_{ij}^{q_3=1} \cdot q_3 + M_{ij}^{x_1=1} \cdot x_1 + M_{ij}^{x_2=1} \cdot x_2$$

$$\mathbf{M}_s = \begin{bmatrix} M_{12} \\ M_{12,half} \\ M_{21} \\ M_{23} \\ M_{23,half} \\ M_{32} \\ M_{34} \\ M_{34,half} \\ M_{43} \\ M_{45} \\ M_{45,half} \\ M_5 \end{bmatrix} = \begin{bmatrix} 2.22408 \\ 0.2206 \\ -1.78289 \\ -1.78289 \\ -0.730335 \\ 0.322215 \\ 0.0249139 \\ 0.822223 \\ 1.61953 \\ 1.61953 \\ -0.261439 \\ -2.14241 \end{bmatrix} [\text{kNm}] \quad \mathbf{M}_c = \begin{bmatrix} M_{12} \\ M_{12,half} \\ M_{21} \\ M_{23} \\ M_{23,half} \\ M_{32} \\ M_{34} \\ M_{34,half} \\ M_{43} \\ M_{45} \\ M_{45,half} \\ M_5 \end{bmatrix} = \begin{bmatrix} -1.68798 \\ -0.298392 \\ 1.0912 \\ 1.0912 \\ 0.880601 \\ 0.670005 \\ 0.867692 \\ -0.473564 \\ -1.81482 \\ -1.81482 \\ 0.117486 \\ 2.04979 \end{bmatrix} [\text{kNm}]$$

$$\text{am } M_{ij} = \sqrt{M_{ijS}^2 + M_{ijC}^2}$$

$$\text{am } \mathbf{M} = \begin{bmatrix} \text{am } M_{12} \\ \text{am } M_{12,half} \\ \text{am } M_{21} \\ \text{am } M_{23} \\ \text{am } M_{23,half} \\ \text{am } M_{32} \\ \text{am } M_{34} \\ \text{am } M_{34,half} \\ \text{am } M_{43} \\ \text{am } M_{45} \\ \text{am } M_{45,half} \\ \text{am } M_{54} \end{bmatrix} = \begin{bmatrix} 2.79 \\ 0.37 \\ 2.09 \\ 2.09 \\ 1.14 \\ 0.74 \\ 0.87 \\ 0.95 \\ 2.43 \\ 2.43 \\ 0.29 \\ 2.97 \end{bmatrix} [\text{kNm}]$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Bending moments (the elastic and damping forces)

$$M_{ij} = M_{ij}^{q_1=1} \cdot q_1 + M_{ij}^{q_2=1} \cdot q_2 + M_{ij}^{q_3=1} \cdot q_3 + M_{ij}^{x_1=1} \cdot x_1 + M_{ij}^{x_2=1} \cdot x_2$$

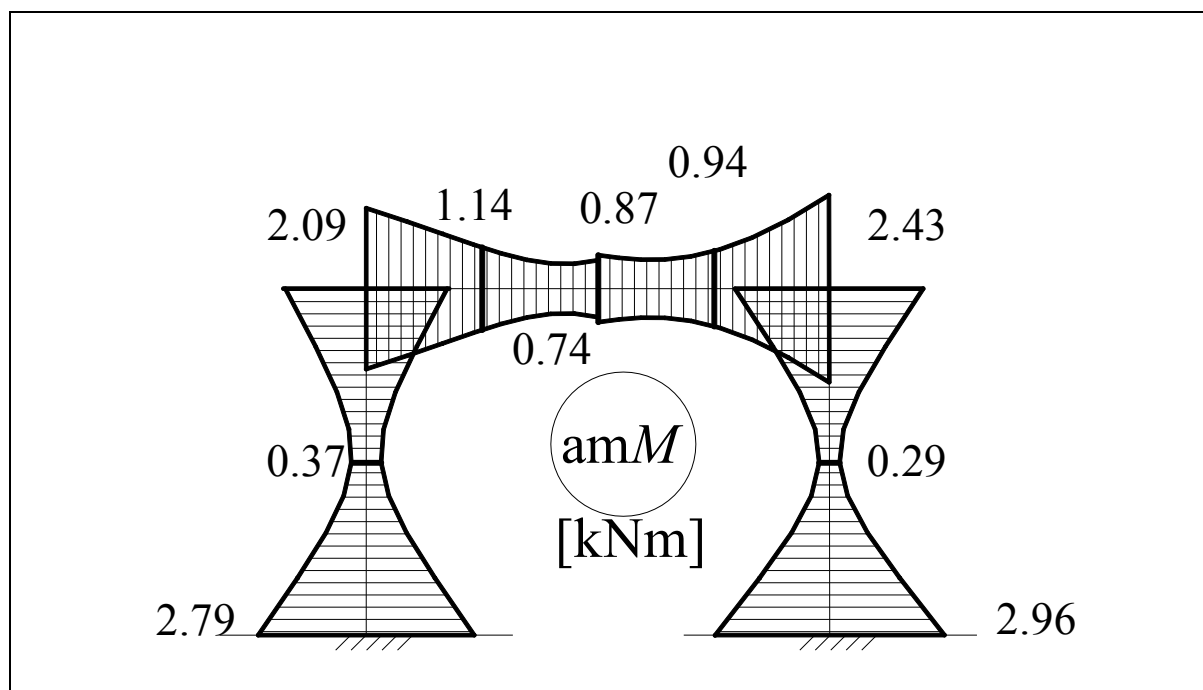
$$\mathbf{M}_s = \begin{bmatrix} M_{12} \\ M_{12,half} \\ M_{21} \\ M_{23} \\ M_{23,half} \\ M_{32} \\ M_{34} \\ M_{34,half} \\ M_{43} \\ M_{45} \\ M_{45,half} \\ M_5 \end{bmatrix} = \begin{bmatrix} 2.56168 \\ 0.280279 \\ -2.00113 \\ -2.00113 \\ -0.906456 \\ 0.188214 \\ -0.148625 \\ 0.916935 \\ 1.9825 \\ 1.9825 \\ -0.284936 \\ -2.55237 \end{bmatrix} [\text{kNm}] \quad \mathbf{M}_c = \begin{bmatrix} M_{12} \\ M_{12,half} \\ M_{21} \\ M_{23} \\ M_{23,half} \\ M_{32} \\ M_{34} \\ M_{34,half} \\ M_{43} \\ M_{45} \\ M_{45,half} \\ M_5 \end{bmatrix} = \begin{bmatrix} -1.24316 \\ -0.254272 \\ 0.734621 \\ 0.734621 \\ 0.734534 \\ 0.734448 \\ 0.872675 \\ -0.309119 \\ -1.49091 \\ -1.49091 \\ 0.0651982 \\ 1.62131 \end{bmatrix} [\text{kNm}]$$

$$\text{am } M_{ij} = \sqrt{M_{ijS}^2 + M_{ijC}^2}$$

$$\text{am } \mathbf{M} = \begin{bmatrix} \text{am } M_{12} \\ \text{am } M_{12,half} \\ \text{am } M_{21} \\ \text{am } M_{23} \\ \text{am } M_{23,half} \\ \text{am } M_{32} \\ \text{am } M_{34} \\ \text{am } M_{34,half} \\ \text{am } M_{43} \\ \text{am } M_{45} \\ \text{am } M_{45,half} \\ \text{am } M_{54} \end{bmatrix} = \begin{bmatrix} 2.85 \\ 0.38 \\ 2.13 \\ 2.13 \\ 1.17 \\ 0.76 \\ 0.89 \\ 0.97 \\ 2.48 \\ 2.48 \\ 0.29 \\ 3.02 \end{bmatrix} [\text{kNm}]$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Bending moments (the elastic forces only)



Bending moments (the elastic and damping forces)

