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THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Dynamics of Structures

Part 1

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THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Matrices

Definition of matrix

A matrix is set of numbers or algebraic expressions arranged in a rectangular form with m rows and n columns. A matrix is denoted by an underlined letter „ $\underline{A}$ ” or a bold letter „ $\mathbf{A}$ ”. The abbreviation „dim” denotes the dimension of matrix. Any element of matrix is described by two subscripts a_{ij} , where the first one, „ i ”, denotes the number of the row and the second, „ j ” - the number of the column. The element a_{ij} is placed on the intersection of i -th row and j -th column.

$$\mathbf{A} = \underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-11} & a_{m-12} & \cdots & a_{m-1n-1} & a_{m-1n} \\ a_{m1} & a_{m2} & \cdots & a_{mn-1} & a_{mn} \end{bmatrix}, \quad \dim \underline{\underline{A}} = m \times n$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

i-th row: $a_{i\bullet}$

j-th column: $a_{\bullet j}$

element: a_{ij}

every element of matrix $\mathbf{A}$: $[A]_{ij} = \forall_{i,j} a_{ij}$

Types of matrices

$\dim \underline{\underline{A}} = m \times n \Rightarrow$ rectangular matrix

$\dim \underline{\underline{B}} = 1 \times n \Rightarrow$ row matrix

$\dim \underline{\underline{C}} = m \times 1 \Rightarrow$ column matrix, vector $\overline{C}$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$\dim \underline{D} = n \times n \Rightarrow$ square matrix

$\dim \underline{E} = m \times n$ and $[E]_{ij} = 0 \Rightarrow$ zero matrix, $\underline{0}, 0_{m \times n}$

$\dim \underline{F} = n \times n$ and $f_{ij} = 0$ for $i \neq j \Rightarrow$ diagonal matrix,

$$\begin{bmatrix} f_{11} & 0 & \cdots & 0 \\ 0 & f_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{nn} \end{bmatrix} = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n \end{bmatrix} = \{f\} = \text{diag}(f_1, f_2, \dots, f_n)$$

$\dim \underline{I} = n \times n$ and $i_{ij} = 0$ for $i \neq j$ and $i_{ii} = 1 \Rightarrow$ identity matrix $\underline{I} = \underline{1}_n = \underline{1}_n = \underline{1} = \underline{1}$

e.g. $\underline{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \underline{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Equality of matrices

$$\underline{A} = \underline{B} \Leftrightarrow [A]_{ij} = [B]_{ij}$$

Scalar multiplication

$$\underline{B} = \alpha \underline{A} \Leftrightarrow [B]_{ij} = \alpha [A]_{ij}; \quad \alpha - \text{scalar}$$

Matrix addition

Matrix addition is possible only if the dimensions of matrices are equal.

$$\underline{C} = \underline{A} + \underline{B} \Leftrightarrow [C]_{ij} = [A]_{ij} + [B]_{ij} \qquad \dim \underline{A} = \dim \underline{B} = \dim \underline{C}$$

Property of matrix addition

$$\underline{A} + \underline{B} = \underline{B} + \underline{A}$$

$$\underline{A} + \underline{B} + \underline{C} = (\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C})$$

$$\underline{A} + \underline{0} = \underline{A}$$

$$\underline{A} - \underline{A} = \underline{0}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Matrix multiplication

$\underline{A} \cdot \underline{B} = \underline{C}$ multiplication is possible if $\dim \underline{C} = m \times p$ where $\dim \underline{A} = m \times n$, $\dim \underline{B} = n \times p$. The result of multiplication is equal to

$$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \dots + a_{in} \cdot b_{nj}$$

The fact that the multiplication $\underline{A} \cdot \underline{B}$ is possible does not mean that the multiplication $\underline{B} \cdot \underline{A}$ is possible.

Example:

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}; \quad \underline{B} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}; \quad \underline{C} = \underline{A} \cdot \underline{B} = ?$$

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C_{31}	C_{32}	C_{33}	C_{34}																		

$$\underline{A} \cdot \underline{B} = \underline{C} = \begin{bmatrix} 9 & 8 & 7 & 6 \\ 14 & 13 & 12 & 11 \\ 8 & 11 & 14 & 17 \end{bmatrix}, \quad \text{but } \underline{B} \cdot \underline{A} \text{ does not exist !}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Hence, the following phrases are used to describe multiplication $\underline{A} \cdot \underline{B}$:

matrix $\underline{A}$ is right multiplied by matrix $\underline{B}$ or matrix $\underline{A}$ is postmultiplied by matrix $\underline{B}$;

matrix $\underline{B}$ is left multiplied by matrix $\underline{A}$ or matrix $\underline{B}$ is premultiplied by matrix $\underline{A}$.

Properties of matrix multiplication

In general the matrix multiplication is not commutative.

$$\underline{A} \cdot \underline{B} \cdot \underline{C} = (\underline{A} \cdot \underline{B}) \cdot \underline{C} = \underline{A} \cdot (\underline{B} \cdot \underline{C})$$

$$\alpha (\underline{A} \cdot \underline{B}) = (\alpha \underline{A}) \cdot \underline{B} = \underline{A} \cdot (\alpha \underline{B})$$

$$(\underline{A} + \underline{B}) \cdot \underline{C} = \underline{A} \cdot \underline{C} + \underline{B} \cdot \underline{C} = \underline{B} \cdot \underline{C} + \underline{A} \cdot \underline{C}$$

$$\underline{C} \cdot (\underline{A} + \underline{B}) = \underline{C} \cdot \underline{A} + \underline{C} \cdot \underline{B} = \underline{C} \cdot \underline{B} + \underline{C} \cdot \underline{A}$$

$$\underline{A} \cdot \underline{0} = \underline{0} \cdot \underline{A} = \underline{0}$$

$$\underline{A} \cdot \underline{I} = \underline{I} \cdot \underline{A} = \underline{A}$$

From the results $\underline{A} \cdot \underline{B} = \underline{0}$ one cannot derive that $\underline{A} = \underline{0}$, $\underline{B} = \underline{0}$ or $\underline{B} \cdot \underline{A} = \underline{0}$.

Matrix multiplication is commutative only for diagonal matrices with the same dimensions:

$$\{a\} \cdot \{b\} = \{b\} \cdot \{a\} = \{ab\} \quad \dim\{a\} = \dim\{b\}$$

Example:

$$\underline{A} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}, \quad \{b\} = \text{diag}(2, 3), \quad \underline{A} \cdot \{b\} = ? \quad \{b\} \cdot \underline{A} = ?$$

$$\underline{A} \cdot \{b\} = \begin{bmatrix} 2 & 9 \\ 10 & 6 \end{bmatrix}. \text{ It is a column multiplication, i.e. the } i\text{-th column of matrix A is multiplied by the}$$

$$i\text{-th element of diagonal } \{b\} - \underline{A} \cdot \{b\} = [a_{\bullet j} \cdot b_j]$$

$$\{b\} \cdot \underline{A} = \begin{bmatrix} 2 & 6 \\ 15 & 6 \end{bmatrix}. \text{ It is a row multiplication, i.e. the } j\text{-th row of matrix A is multiplied by the } j\text{-th}$$

$$\text{element of diagonal } \{b\} - \{b\} \cdot \underline{A} = [b_j \cdot a_{i \bullet}].$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

Matrix transposition

$$\underline{A} = \underline{B}^T \Leftrightarrow \dim \underline{A} = m \times n, \Rightarrow \dim \underline{B} = n \times m \text{ and } [A]_{ij} = [B]_{ji}$$

Example:

$$\underline{B} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}; \quad \underline{B}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} = \underline{A}$$

$$\underline{A} = \underline{B}^T \Leftrightarrow \underline{B} = \underline{A}^T$$

$$\underline{A} = (\underline{A}^T)^T$$

$$(\underline{A} + \underline{B} + \underline{C})^T = \underline{A}^T + \underline{B}^T + \underline{C}^T$$

$$(\underline{A} \cdot \underline{B} \cdot \underline{C})^T = \underline{C}^T \cdot \underline{B}^T \cdot \underline{A}^T$$

If $\underline{A} = \underline{A}^T$ then $\underline{A}$ is a symmetric matrix, $[A]_{ij} = [A]_{ji}$

If $\underline{A} = -\underline{A}^T$ then $\underline{A}$ is a skew-symmetric matrix, $[A]_{ij} = -[A]_{ji}$, $[A]_{ii} = 0$

Every diagonal matrix is also a symmetric matrix $\{a\}^T = \{a\}$.

Trace of matrix

$\dim \underline{A} = n \times n$,

$$\text{tr } \underline{A} = \sum_{k=1}^n a_{kk} = a_{11} + a_{22} + \dots + a_{nn}$$

$$\text{tr } \underline{A}^T = \text{tr } \underline{A}$$

$$\text{tr } (\underline{A} + \underline{B} + \underline{C}) = \text{tr } \underline{A} + \text{tr } \underline{B} + \text{tr } \underline{C}$$

$$\text{tr } (\alpha \underline{A}) = \alpha (\text{tr } \underline{A})$$

$$\text{tr } (\underline{A} \cdot \underline{B}) = \text{tr } (\underline{B} \cdot \underline{A})$$

$$\text{tr } (\underline{A} \cdot \underline{B} \cdot \underline{C}) = \text{tr } (\underline{B} \cdot \underline{C} \cdot \underline{A}) = \text{tr } (\underline{C} \cdot \underline{A} \cdot \underline{B}) \neq \text{tr } (\underline{A} \cdot \underline{C} \cdot \underline{B})$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Example:

$$\text{tr} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 5 & 2 & 1 & 2 \\ 2 & 3 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = 1 + 2 + 0 + 4 = 7$$

Determinant of matrix

A determinant of matrix $\underline{A}$ is denoted by „det $\underline{A}$ “. The determinant is a scalar and is defined solely for the square matrix $\dim \underline{A} = n \times n$. The easiest way to define the determinant is recursion. For a matrix with dimensions 1×1 , the determinant is equal in value to the only element of this matrix. For a matrix with dimensions 2×2 , the determinant is equal to:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For a matrix with dimensions 3×3 , the rule of Sarrus can be used. Two first columns should be written on the right side of the matrix (or two first rows -- under the matrix). The product of the elements crossed with a blue line are added, and the ones crossed with a red line are subtracted:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

For matrix $\underline{A}$ with dimensions $\dim \underline{A} = n \times n$ det $\underline{A}$ is defined by determinants of matrices with dimensions $(n-1) \times (n-1)$. The minor $\underline{M}_{ij}$ is a matrix obtained from matrix $\underline{A}$ by erasing the i -th row and the j -th column. The cofactor A_{ij} of the element a_{ij} is a determinant of the minor $\underline{M}_{ij}$ multiplied by $(-1)^{i+j}$.

$$A_{ij} = (-1)^{i+j} \times \det \underline{M}_{ij}$$

The determinant may be calculated by application of the Laplace expansion along any i -th row



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$\det \underline{A} = \sum_{k=1}^n (-1)^{i+k} a_{ik} \det \underline{M}_{ik} = \sum_{k=1}^n a_{ik} A_{ik}$$

or along any j -th row

$$\det \underline{A} = \sum_{k=1}^n (-1)^{k+j} a_{kj} \det \underline{M}_{kj} = \sum_{k=1}^n a_{kj} A_{kj}$$

Example

$$\det \begin{bmatrix} 1 & 3 & 0 & 1 \\ 5 & 2 & 1 & 2 \\ 2 & 3 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = ?$$

Using the Laplace expansion along the 3rd column and the rule of Sarrus, the determinant can be easily calculated:

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & 1 \\ 5 & 2 & 1 & 2 \\ 2 & 3 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} &= 1 \times (-1)^{2+3} \det \begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix} + 3 \times (-1)^{4+3} \det \begin{bmatrix} 1 & 3 & 1 \\ 5 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix} = \\ &= -1 \times (1 \times 3 \times 4 + 3 \times 3 \times 1 + 1 \times 2 \times 2 - 1 \times 3 \times 1 - 1 \times 2 \times 3 - 3 \times 2 \times 4) + \\ &- 3 \times (1 \times 2 \times 3 + 3 \times 2 \times 2 + 1 \times 5 \times 3 - 1 \times 2 \times 2 - 1 \times 2 \times 3 - 3 \times 5 \times 3) = 8 + 66 = 74 \end{aligned}$$

Adding a column/row multiplied by any number to another column/row does not change the value of the determinant. This property makes it possible to calculate the determinant easily. In the example below:

- (1) the 2nd row multiplied by -3 is added to the 4th row. After this, there is only one nonzero element in the 3rd column
- (2) the Laplace expansion is applied along the 3rd column
- (3) the 1st row multiplied by -2 is added to the 2nd row and the 1st row multiplied by 14 is added to the 3rd row.
- (4) the Laplace expansion is applied along the 1st column and the determinant can be calculated easily for matrix 2×2.



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & 1 \\ 5 & 2 & 1 & 2 \\ 2 & 3 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} &\stackrel{(1)}{=} \det \begin{bmatrix} 1 & 3 & 0 & 1 \\ 5 & 2 & 1 & 2 \\ 2 & 3 & 0 & 3 \\ -14 & -4 & 0 & -2 \end{bmatrix} \stackrel{(2)}{=} 1 \times (-1)^{2+3} \det \begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 3 \\ -14 & -4 & -2 \end{bmatrix} \stackrel{(3)}{=} \\ &= -1 \times \det \begin{bmatrix} 1 & 3 & 1 \\ 0 & -3 & 1 \\ 0 & 38 & 12 \end{bmatrix} \stackrel{(4)}{=} -1 \times (-1)^{1+1} \det \begin{bmatrix} -3 & 1 \\ 38 & 12 \end{bmatrix} = -1 \times (-3 \times 12 - 1 \times 38) = 74 \end{aligned}$$

Some properties of determinant:

$$\det \underline{A}^T = \det \underline{A}$$

$$\det (\alpha \underline{A}) = \alpha^n (\det \underline{A})$$

$$\det (\underline{A} \cdot \underline{B} \cdot \underline{C}) = \det \underline{A} \cdot \det \underline{B} \cdot \det \underline{C}$$

$$\det \{a\} = a_1 \cdot a_2 \cdot \dots \cdot a_n$$

$\underline{A}$ is called a singular matrix or degenerate matrix if $\det \underline{A} = 0$.

$\underline{A}$ is called a non-singular matrix (nondegenerate matrix or invertible matrix) if $\det \underline{A} \neq 0$.

Inversion of matrix

A square matrix $\underline{B}$ with the same dimensions as matrix $\underline{A}$ is called inverse to it when:

$$\underline{B} \cdot \underline{A} = \underline{A} \cdot \underline{B} = \underline{I}$$

The symbol $\underline{A}^{-1}$ is used for a matrix inverse to $\underline{A}$.

Using the properties of the determinant:

$$\underline{A}^2 = \underline{A} \cdot \underline{A}, \quad \underline{A}^3 = \underline{A} \cdot \underline{A} \cdot \underline{A}$$

$$\det (\underline{A}^n) = (\det \underline{A})^n$$

$$\underline{A}^{-1} \cdot \underline{A} = \underline{A} \cdot \underline{A}^{-1} = \underline{I}$$

$$\det (\underline{A} \cdot \underline{A}^{-1}) = \det \underline{A} \cdot \det \underline{A}^{-1} = \det \underline{I} = 1$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$\det \underline{A}^{-1} = 1/\det \underline{A} \rightarrow \underline{A}^{-1}$ exist only if $\det \underline{A} \neq 0$ ($\underline{A}$ is a non-singular matrix).

Matrix $\underline{A}^{-1}$ can be derived using formula:

$$\underline{A}^{-1} = \text{adj } \underline{A} / \det \underline{A}$$

where $\text{adj } \underline{A}$ – is an adjugate (or adjoint) matrix i.e. a transposed matrix of cofactors A_{ij} .

$$\text{adj } \underline{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Example:

$$\underline{A} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}, \det \underline{A} = 9+0+20-15-12-0 = 2$$

$$\text{adj } \underline{A} = \begin{bmatrix} \det \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix} & -\det \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} & \det \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \\ -\det \begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix} & \det \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} & -\det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \\ \det \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} & -\det \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} & \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \end{bmatrix}^T =$$

$$= \begin{bmatrix} 9 & -6 & 1 \\ 4 & -2 & 0 \\ -15 & 10 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & 4 & -15 \\ -6 & -2 & 10 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\underline{A}^{-1} = \text{adj } \underline{A} / \det \underline{A} = \begin{bmatrix} 4,5 & 2 & -7,5 \\ -3 & -1 & 5 \\ 0,5 & 0 & -0,5 \end{bmatrix}$$

Any matrix $\underline{A}$ could be transformed into an identity matrix $\underline{I}$ by performing the following transformations (multiplication of the row/column by a scalar, adding rows/columns).

By applying the same transformations to the identity matrix one can obtain a matrix inverse to $\underline{A}$.



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

$$\underline{A} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}, \underline{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{cases} 2^{\text{nd}} \text{ row} = \text{row} 2^{\text{nd}} - 1^{\text{st}} \text{ row} \times 2 \\ 3^{\text{rd}} \text{ row} = 3^{\text{rd}} \text{ row} - 1^{\text{st}} \text{ row} \end{cases}$$

$$\underline{A}_{(\text{step1})} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -10 \\ 0 & 0 & -2 \end{bmatrix}, \underline{I}_{(\text{step1})} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{cases} 2^{\text{nd}} \text{ row} = 2^{\text{nd}} \text{ row} / \text{element}_{22} \\ 1^{\text{st}} \text{ row} = 1^{\text{st}} \text{ row} - 2^{\text{nd}} \text{ row} \times 2 \end{cases}$$

$$\underline{A}_{(\text{step2})} = \begin{bmatrix} 1 & 0 & -15 \\ 0 & 1 & 10 \\ 0 & 0 & -2 \end{bmatrix}, \underline{I}_{(\text{step2})} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{cases} 3^{\text{rd}} \text{ row} = 3^{\text{rd}} \text{ row} / (-2) \\ 1^{\text{st}} \text{ row} = 1^{\text{st}} \text{ row} + 3^{\text{rd}} \text{ row} \times 15 \\ 2^{\text{nd}} \text{ row} = 2^{\text{nd}} \text{ row} - 3^{\text{rd}} \text{ row} \times 10 \end{cases}$$

$$\underline{A}_{(\text{step3})} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I}, \underline{I}_{(\text{step3})} = \begin{bmatrix} 4,5 & 2 & -7,5 \\ -3 & -1 & 5 \\ 0,5 & 0 & -0,5 \end{bmatrix} = \underline{A}^{-1}$$

Verification of: $\underline{A}^{-1} \cdot \underline{A} = ?$, $\underline{A} \cdot \underline{A}^{-1} = ?$

$$\begin{bmatrix} 4,5 & 2 & -7,5 \\ -3 & -1 & 5 \\ 0,5 & 0 & -0,5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4,5 & 2 & -7,5 \\ -3 & -1 & 5 \\ 0,5 & 0 & -0,5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Vectors

Matrices with only one column are often called vectors. Vectors are denoted by overlining

$$\bar{x} = \text{col}(x_1, x_2, \dots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \dots, x_n]^T$$

Two vectors are orthogonal if $\bar{x} \cdot \bar{y}^T = \bar{x}^T \cdot \bar{y} = 0$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Two vectors are orthogonal with weight $\underline{A}$ if $\bar{x} \cdot \underline{A} \cdot \bar{y}^T = \bar{x}^T \cdot \underline{A} \cdot \bar{y} = 0$.

The normalization of a vector $\bar{x}$ is a result of dividing elements of this vector by number $\|\bar{x}\|$. The most common method of normalization is dividing a vector 1) by its maximum absolute value or 2) by the Euclidean norm.

Example:

$$\bar{a} = \begin{bmatrix} 4 \\ 4 \\ -7 \end{bmatrix}$$

$$1) \quad \|\bar{x}\| = \max |x_i| \quad \text{maximum norm} \quad \|\bar{a}\| = 7$$

$$2) \quad \|\bar{x}\| = \sqrt{\bar{x}^T \cdot \bar{x}} \quad \text{Euclidean norm} \quad \|\bar{a}\| = \sqrt{4^2 + 4^2 + (-7)^2} = 9$$

Normalized vector $\bar{x}_{\text{norm}} = \bar{x} / \|\bar{x}\|$

$$1) \quad \bar{a}_{\text{norm}} = \text{col}\left(\frac{4}{7}, \frac{4}{7}, -1\right) \quad -1 \leq \bar{a}_{\text{norm},i} \leq 1$$

$$2) \quad \bar{a}_{\text{norm}} = \text{col}\left(\frac{4}{9}, \frac{4}{9}, -\frac{7}{9}\right) \quad \|\bar{a}_{\text{norm}}\| = 1$$

System of linear equations

The system of n linear equations with n unknowns $(x_1, x_2, \dots, x_n)$ could be written down with matrix notation

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = y_n \end{cases} \longrightarrow \underline{A} \cdot \bar{x} = \bar{y}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

A system of linear equations is called a homogenous system if $\bar{y} = 0$, and a nonhomogenous system if $\bar{y} \neq 0$. The existence of a solution depends on the type of the system (homogenous or not) and on the properties of matrix $\underline{A}$ (if it is a singular matrix or not).

1. nonhomogenous system ($\bar{y} \neq 0$)

1.1. $\det A \neq 0 \Rightarrow$ solution exists $\bar{x} = \underline{A}^{-1} \cdot \bar{y}$

1.2. $\det A = 0 \Rightarrow$ solution does not exist

2. homogenous system ($\bar{y} = 0$)

2.1. $\det A \neq 0 \Rightarrow$ solution exists and is trivial $\bar{x} = \bar{0}$

2.2. $\det A = 0 \Rightarrow$ there is an infinite number of solutions

$$\bar{x} = \text{adj } \underline{A} \cdot \bar{v}, \quad \bar{v} - \text{any arbitrary vector,}$$

for $\bar{v} = \text{col}(0, 0, \dots, 1, \dots, 0)$ the solution is any non-zero column of matrix $\text{adj } \underline{A}$; generally the solution is a linear combination of columns of matrix $\text{adj } \underline{A}$.

Eigenproblem

Numbers λ_i are eigenvalues of a square matrix $\underline{A}$ if a nonzero vector $\bar{w}_i$ exists which fulfils the equation $\underline{A} \cdot \bar{w}_i = \lambda_i \cdot \bar{w}_i$. Vectors $\bar{w}_i$ are called eigenvectors.

$$\underline{A} \cdot \bar{w} = \lambda \cdot \bar{w}$$

$$\underline{A} \cdot \bar{w} - \lambda \cdot \bar{w} = \bar{0}$$

$$(\underline{A} - \lambda \underline{I}) \cdot \bar{w} = \bar{0}; \quad \bar{w} \neq \bar{0} \Leftrightarrow \det(\underline{A} - \lambda \underline{I}) = 0$$

$$\det \begin{bmatrix} a_{11} - \lambda & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} - \lambda & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} - \lambda \end{bmatrix} = 0$$

$$\det(\underline{A} - \lambda \underline{I}) = 0 \Leftrightarrow \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_2 \lambda^2 + \alpha_1 \lambda^1 + \alpha_0 \lambda^0 = 0 - \text{characteristic equation}$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

The solutions of this characteristic equation $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$ are eigenvalues.

Example:

Calculate eigenvalues of matrix $\underline{A} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$,

characteristic equation: $\det \begin{bmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{bmatrix} = 0 \rightarrow (3-\lambda)(6-\lambda) - 4 = 0,$

$$\lambda^2 - 9\lambda + 14 = 0 \rightarrow \text{solutions } \lambda_1 = 2, \lambda_2 = 7$$

The eigenvector is any non-zero column of the matrix adjoint to matrix $(\underline{A} - \lambda_i \underline{I})$

$$\lambda_1 = 2; \underline{B}_1 = (\underline{A} - \lambda_1 \underline{I}) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \text{adj } \underline{B}_1 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \xrightarrow{\text{normalization}} \begin{bmatrix} 1 & 1 \\ -1/2 & -1/2 \end{bmatrix}, \bar{\underline{w}}_1 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix},$$

$$\lambda_2 = 7; \underline{B}_2 = (\underline{A} - \lambda_2 \underline{I}) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}, \text{adj } \underline{B}_2 = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \xrightarrow{\text{normalization}} \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix}, \bar{\underline{w}}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Any product of $\alpha \bar{\underline{w}}_i$ is also an eigenvector, where α - any number.

$$\bar{\underline{w}}_1 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \bar{\underline{w}}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \underline{W} = [\bar{\underline{w}}_1 \quad \bar{\underline{w}}_2] = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix},$$

A matrix $\underline{W}$, the i -th column of which is equal to $\bar{\underline{w}}_i$, is called an eigenmatrix.

Using system “*Mathematica*” for matrix calculations

In the system *Mathematica*, matrices are defined as lists of lists e.g. lists of rows. The system gives output in the same form. In order to see the matrix in the form of a table, one needs to use the command “TraditionalForm” or “//MatrixForm”.

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

```
In[1]:= A = {{1, 2, 5}, {2, 3, 0},
            {1, 2, 3}}
```

```
Out[1]= {{1, 2, 5}, {2, 3, 0}, {1, 2, 3}}
```

```
In[2]:= A
```

```
Out[2]= {{1, 2, 5}, {2, 3, 0}, {1, 2, 3}}
```

```
In[3]:= A // MatrixForm
```

```
Out[3]/MatrixForm=

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

```

```
In[4]:= TraditionalForm[A]
```

```
Out[4]/TraditionalForm=

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

```

```
In[5]:= v = {{1}, {2}, {3}}
```

```
Out[5]= {{1}, {2}, {3}}
```

```
In[6]:= MatrixForm[v]
```

```
Out[6]/MatrixForm=

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

```

In order to transpose the matrix, the command “Transpose” should be used. The transposition of a vector with one column yields a matrix with one row only.

```
In[7]:= Transpose[A]
```

```
Out[7]= {{1, 2, 1}, {2, 3, 2}, {5, 0, 3}}
```

```
In[8]:= TraditionalForm[Transpose[A]]
```

```
Out[8]/TraditionalForm=

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 5 & 0 & 3 \end{pmatrix}$$

```

```
In[9]:= Transpose[v] // MatrixForm
```

```
Out[9]/MatrixForm=

$$(1 \ 2 \ 3)$$

```

For matrices, the multiplication symbol “.” (dot) is used.

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

```
In[10]:= A.A // MatrixForm
```

```
Out[10]/MatrixForm=
```

$$\begin{pmatrix} 10 & 18 & 20 \\ 8 & 13 & 10 \\ 8 & 14 & 14 \end{pmatrix}$$

```
In[11]:= A.v // MatrixForm
```

```
Out[11]/MatrixForm=
```

$$\begin{pmatrix} 20 \\ 8 \\ 14 \end{pmatrix}$$

```
In[12]:= Transpose[v].A // MatrixForm
```

```
Out[12]/MatrixForm=
```

$$(8 \ 14 \ 14)$$

The command “Det” is used for calculating the determinant,

“Tr” is used for calculating the trace

“Inverse” is used for calculating the inversion of the matrix.

```
In[13]:= Det[A]
```

```
Out[13]=
```

2

```
In[14]:= TraditionalForm[Inverse[A]]
```

```
Out[14]/TraditionalForm=
```

$$\begin{pmatrix} \frac{9}{2} & 2 & -\frac{15}{2} \\ -3 & -1 & 5 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

```
In[15]:= Tr[A]
```

```
Out[15]=
```

7

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

```
In[17]:= TraditionalForm[Inverse[A].A]
```

```
Out[17]//TraditionalForm=
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[18]:= TraditionalForm[IdentityMatrix[3]]
```

```
Out[18]//TraditionalForm=
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[19]:= DiagonalMatrix[{1, 2, 3}] // MatrixForm
```

```
Out[19]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The command “IdentityMatrix” can be used in order to declare an identity matrix, and the command “DiagonalMatrix” – to declare a diagonal matrix. The command “Eigenvalues” yields the eigenvalues of the matrix, arranged from the greatest to the smallest. The command “Eigenvectors” yields the eigenvectors of the matrix. These eigenvectors are arranged in rows! The first row corresponds to the greatest eigenvalue. The command “Eigensystem” combines two previous commands.

```
In[26]:= AA = {{3, 2}, {2, 6}}; AA // MatrixForm
```

```
Out[26]//MatrixForm=
```

$$\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$$

```
In[27]:= Eigenvalues[AA]
```

```
Out[27]=
```

$$\{7, 2\}$$

```
In[28]:= Eigenvectors[AA] // MatrixForm
```

```
Out[28]//MatrixForm=
```

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

```
In[29]:= Eigensystem[AA] // MatrixForm
```

```
Out[29]//MatrixForm=
```

$$\begin{pmatrix} 7 & 2 \\ \{1, 2\} & \{-2, 1\} \end{pmatrix}$$

THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCLAW UNIVERSITY OF TECHNOLOGY

Problems

The following matrices are given:

$$\underline{A} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad \underline{E} = \begin{bmatrix} 2 & 4 \\ 4 & 1 \\ 6 & 2 \end{bmatrix},$$

$$\underline{F} = \begin{bmatrix} 3 & 7 & 1 \\ -3 & 0 & 1 \end{bmatrix}, \quad \underline{G} = \begin{bmatrix} 1 & 2 & 1 & 5 \\ -3 & 2 & 1 & 0 \\ 4 & 2 & 7 & 1 \\ 0 & 2 & 2 & 3 \end{bmatrix}, \quad \underline{U} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underline{V} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix},$$

$$\underline{L} = \text{diag} (1, 2), \quad \underline{M} = \text{diag} (1, 2, 1), \quad \underline{N} = \text{diag} (3, 2, 4, 1).$$

- Evaluate: (a) $\underline{A} + \underline{A}$, (b) $\underline{A} \cdot \underline{A}$, (c) $\underline{A} \cdot \underline{B}$, (d) $\underline{B} \cdot \underline{C}^T$, (e) $\underline{E} \cdot \underline{F}$, (f) $\underline{D} \cdot \underline{V}$, (g) $\underline{X}^T \cdot \underline{G}$, (h) $\underline{D} \cdot \underline{M}$, (i) $\underline{N} \cdot \underline{G}$, (j) $\underline{V}^T \cdot \underline{V}$, (k) $\underline{V} \cdot \underline{V}^T$.
- Evaluate: (a) $\text{tr } \underline{G}$, (b) $\det \underline{C}$, (c) $\det \underline{D}$, (d) $\det \underline{G}$, (e) $\det (\underline{L} \cdot \underline{C} \cdot \underline{L})$.
- Evaluate: (a) $\text{adj } \underline{B}$, (b) $\text{adj } \underline{N}$, (c) $\text{adj } (\underline{U} \cdot \underline{U}^T)$.
- Evaluate eigenvalues and eigenvectors of matrices: (a) $\underline{A}$, (b) $\underline{D}$, (c) $\underline{G}$.
- Solve the system of equations:

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + 3y + 4z = 0 \\ 2x + 1y = 0 \end{cases}$$

- Find the value of λ for which a nontrivial solution $\bar{\underline{V}}$ of these equations exists:

$$(\underline{K} - \lambda \cdot \underline{B}) \cdot \bar{\underline{V}} = \bar{\underline{0}},$$

Find vectors $\bar{\underline{V}}$ and normalize them

$$(a) \quad \underline{K} = \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \text{ find the solution for } m \text{ and } k \text{ or assume } k=1, m=1,$$

$$(b) \quad \underline{K} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, k=1, m=1,$$



Answers to problems

$$1. (a) \underline{A} + \underline{A} = \begin{bmatrix} 4 & 4 \\ 4 & 12 \end{bmatrix}, (b \div d) \underline{A} \cdot \underline{A} = \underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{C}^T = \begin{bmatrix} 8 & 16 \\ 16 & 40 \end{bmatrix},$$

$$(e) \underline{E} \cdot \underline{F} = \begin{bmatrix} -6 & 14 & 6 \\ 9 & 28 & 5 \\ 12 & 42 & 8 \end{bmatrix}, (f) \underline{D} \cdot \underline{V} = \begin{bmatrix} 11 \\ 24 \\ 11 \end{bmatrix},$$

$$(g) \underline{X}^T \cdot \underline{G} = [9 \quad 20 \quad 35 \quad 17]$$

$$(h) \underline{D} \cdot \underline{M} = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 10 & 1 \\ 3 & 2 & 2 \end{bmatrix}, (i) \underline{N} \cdot \underline{G} = \begin{bmatrix} 3 & 6 & 3 & 15 \\ -6 & 4 & 2 & 0 \\ 16 & 8 & 28 & 4 \\ 0 & 2 & 2 & 3 \end{bmatrix},$$

$$(j) \underline{V}^T \cdot \underline{V} = [14], (k) \underline{V} \cdot \underline{V}^T = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 9 & 3 \\ 2 & 3 & 1 \end{bmatrix},$$

$$2. (a) \text{tr } \underline{G} = 13, (b) \det \underline{C} = 8 (c) \det \underline{D} = -34 (d) \det \underline{G} = 26$$

$$(e) \det (\underline{L} \cdot \underline{C} \cdot \underline{L}) = \det \begin{bmatrix} 2 & 4 \\ 4 & 24 \end{bmatrix} = 32,$$

$$3. (a) \text{adj } \underline{B} = \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}, (b) \text{adj } \underline{N} = \text{diag} (8, 12, 6, 24)$$

$$(c) \text{adj} (\underline{U} \cdot \underline{U}^T) = \text{adj} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$4. (a) A : \lambda_1 = 4 - 2\sqrt{2} = 1,17157, \lambda_2 = 4 + 2\sqrt{2} = 6,82843,$$

$$5. W = \begin{bmatrix} -2 + 2\sqrt{2} & 2 \\ 2 & 2 - 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0,4142 & 1 \\ 1 & -0,4142 \end{bmatrix}$$

$$6. D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix}; D(\lambda) = D - I \cdot \lambda; \det D(\lambda) = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 1 \\ 3 & 1 & 2-\lambda \end{bmatrix} = 0$$

$$7. -34 + \lambda + 8\lambda^2 - \lambda^3 = 0 \rightarrow \lambda_1 = -1,904 \quad \lambda_2 = 2,37031 \quad \lambda_3 = 7,53369$$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

8. eigenmatrix $\underline{W} = \begin{bmatrix} 1 & -0,440 & 0,518 \\ -0,486 & 1 & 1 \\ -0,644 & -0,868 & 0,462 \end{bmatrix};$

9. eg. $\text{adj } D(\lambda_3) = \text{adj} \begin{bmatrix} -6,534 & 2 & 3 \\ 4 & -2,534 & 1 \\ 3 & 1 & -5,534 \end{bmatrix} =$

$$\begin{bmatrix} 13,023 & 25,136 & 11,602 \\ 14,068 & 27,159 & 12,534 \\ 9,602 & 18,534 & 8,557 \end{bmatrix}^T = \begin{bmatrix} 13,023 & 14,068 & 9,602 \\ 25,136 & 27,159 & 18,534 \\ 11,602 & 12,534 & 8,557 \end{bmatrix} \xrightarrow{\text{columns' normalization}}$$

$$\begin{bmatrix} 0,518 & 0,518 & 0,518 \\ 1 & 1 & 1 \\ 0,462 & 0,462 & 0,462 \end{bmatrix}$$

10. $\underline{G} = \begin{bmatrix} 1 & 2 & 1 & 5 \\ -3 & 2 & 1 & 0 \\ 4 & 2 & 7 & 1 \\ 0 & 2 & 2 & 3 \end{bmatrix},$

The characteristic equation $26 - 123\lambda + 51\lambda^2 - 13\lambda^3 + \lambda^4 = 0$

$\lambda_1 = 0,232489$; $\lambda_2 = 2,01665 - 2,95587i$; $\lambda_3 = 2,01665 + 2,95587i$; $\lambda_4 = 8,73422$.

Eigenmatrix $\underline{W} = \begin{bmatrix} 0,425 & 1,704 + 1,288i & 1,705 - 1,288i & 0,351 \\ 1 & 1,033 - 2,252i & 1,033 + 2,252i & -0,008 \\ -0,492 & -1,524 + 0,774i & -1,524 - 0,774i & 1 \\ -0,367 & 1 & 1 & 0,346 \end{bmatrix},$

11. $\begin{cases} x + 2y + 3z = 0 \\ 2x + 3y + 4z = 0 \\ 2x + y = 0 \end{cases}$

System matrix $\underline{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 1 & 0 \end{bmatrix}$



THE DEVELOPMENT OF THE POTENTIAL AND ACADEMIC PROGRAMMES OF WROCŁAW UNIVERSITY OF TECHNOLOGY

$$\text{adj } \underline{A} = \begin{bmatrix} -4 & 3 & -1 \\ 8 & -6 & 2 \\ -4 & 3 & -1 \end{bmatrix} \xrightarrow{\text{columns' normalization}} \begin{bmatrix} -0,5 \\ 1 \\ -0,5 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -0,5 \\ 1 \\ -0,5 \end{bmatrix}$$

α – any number

$$12. (a) \quad \underline{K} = \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix},$$

$$\det(\underline{K} - \lambda \cdot \underline{B}) = 0 \rightarrow k^2 - 3 k m \lambda + m^2 \lambda^2 = 0$$

$$\lambda_1 = (3 k - \sqrt{5} k) / 2m = 0,381 k/m \quad \lambda_2 = (3 k + \sqrt{5} k) / 2m = 2,618 k/m$$

$$\underline{W} = \begin{bmatrix} 2k + \frac{1}{2}(-3k + \sqrt{5}k) & k \\ k & k + \frac{1}{2}(-3k - \sqrt{5}k) \end{bmatrix} = \begin{bmatrix} 1,618 & 1 \\ 1 & -1,618 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -0,618 \\ 0,618 & 1 \end{bmatrix}; \quad \bar{v} - \text{any column from } \underline{W}.$$

$$(b) \quad \underline{K} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, k=1, m=1,$$

$$1 - 6\lambda + 5\lambda^2 - \lambda^3 = 0 \rightarrow \lambda_1 = 0,198 \quad \lambda_2 = 1,555 \quad \lambda_3 = 3,247$$

$$\bar{v} - \text{any column from } \begin{bmatrix} 1 & -0,802 & -0,445 \\ 0,802 & 0,445 & 1 \\ 0,445 & 1 & -0,802 \end{bmatrix}$$